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Some Results on Regularity, Self-Similarity and Asymptotic Behavior for Nonlinear Diffusion Equations

Memoria para optar al titulo de

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por

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Introducción

Entre todas las áreas de las matemáticas, las Ecuaciones en Derivadas Parciales es una de las que se usan con frecuencia para describir modelos de las ciencias aplicadas, como la Física, la Química, Ingeniería, Procesamiento de Imágenes y Biología. Además, debido a la gran complejidad de la teoría matemática, las Ecuaciones en Derivadas Parciales han motivado y estimulado el estudio de otras áreas de las matemáticas, que hoy son muy importantes y muy estudiadas, como Análisis Armónico, Análisis Numérico, Análisis Funcional Abstracto y Teoría de Operadores.

Ecuaciones de Difusión No Lineal. Propiedades generales

En esta memoria estudiamos un tipo particular de ecuaciones en derivadas parciales, que se conocen (debido a su relevancia física, explicada abajo) como Ecuaciones de Difusión No Lineal. Esta clase de ecuaciones tiene como prototipo los siguientes dos modelos importantes:

$$u_t = \Delta u^m, \tag{0.1}$$

conocida como la Ecuación de los Medios Porosos (abreviado PME), y

$$u_t = \Delta_p u, \tag{0.2}$$

conocida como la ecuación de evolución p-Laplaciana (abreviada PLE), donde

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

Estas ecuaciones son dos de los representantes más sencillos de la clase de ecuaciones no lineales de evolución de tipo parabólico. Ellas surgen de muchos fenómenos de las ciencias aplicadas que han motivado, junto con su riqueza en propiedades matemáticas poco usuales, su extenso estudio en las últimas décadas. Además, aunque fomalmente se pueden ver como variaciones no lineales de la clásica Ecuación del Calor (abreviada HE),

$$u_t = \Delta u, \tag{0.3}$$

que se puede obtener de la PME poniendo $m = 1$ y de la PLE con $p = 2$, sus propiedades matemáticas son muy diferentes de las de la HE. Además, exponentes diferentes m y p dan lugar a propiedades muy diferentes de las soluciones; en particular, hay varios exponentes críticos donde el comportamiento de las soluciones cambia de una forma esencial.

Ambas ecuaciones, la PME y la PLE, son ecuaciones de evolución no lineal, formalmente de tipo parabólico. Si las escribimos en forma de divergencia

$$u_t = \mathrm{div}(A(u, Du)\nabla u),$$

observamos que el coeficiente de difusión $A(u, Du)$ es $A(u) = mu^{m-1}$ (bajo la restricción $u \geq 0$) para la PME, y $A(u) = |\nabla u|^{p-2}$ para la PLE. Se observa así que ambas ecuaciones pueden ser degeneradas o singulares. Más precisamente, la PME con $m > 1$ es estrictamente parabólica sólo donde $u \neq 0$, y en los puntos donde u se anula, la ecuación es degenerada para $m > 1$ (dado que $A(u) = 0$ en este caso) y es singular para $m < 1$ (dado que $A(u) = \infty$ en este caso). Lo mismo sucede en la PLE, pero ahí el término influente es el módulo del gradiente (decimos que la PLE es el prototipo de las ecuaciones de difusión con dependencia en el gradiente). En los puntos donde $\nabla u = 0$, la PLE degenera para $p > 2$ y es singular para $p < 2$. Por ello, como los rangos de exponentes más importantes históricamente y más estudiados son $m > 1$, respectivamente $p > 2$, éstas ecuaciones se llaman *parabólicas degeneradas*.

El cambio entre los dos rangos $m > 1$ y $m < 1$ en la PME (el último siendo conocido en la literatura como la *Ecuación de difusión rápida*, abreviada FDE), y entre los rangos $p > 2$ y $p < 2$ (el segundo llamado en lo siguiente la *Ecuación de difusión rápida p-Laplaciana* por similaridad con la FDE) se refleja fuertemente en las propiedades básicas de las soluciones. Vamos a precisar esto en la subsección dedicada a las propiedades especiales de las dos ecuaciones.

Modelos y aplicaciones

Ambas ecuaciones PME y PLE aparecen en la descripción de diferentes fenómenos naturales. Vamos a presentar en breve algunas de sus aplicaciones prácticas más importantes o conocidas.

Para la PME estándar ($m > 1$), probablemente la aplicación más conocida (que justifica el nombre de la ecuación) es la descripción del flujo de un gas isentrópico a través de un medio poroso. Esto ha sido modelizado independientemente por Muskat [106] y Leibenzon [101], con una deducción basada en la ley de Darcy. Éste es un modelo muy conocido, pero dado que se puede considerar como el punto de partida para el desarrollo de la teoría, lo voy a presentar de manera esquemática aquí. El modelo se puede formalizar en términos de las variables densidad (notada por ρ), presión (notada por p) y campo de velocidades (notado por $\mathbf{V}$), que se relacionan por las siguientes leyes físicas:

(i) La ecuación de continuidad en la mecánica de fluidos:

$$\varepsilon\rho_t + \mathrm{div}(\rho\mathbf{V}) = 0, \tag{0.4}$$

donde $\varepsilon \in (0, 1)$ es la porosidad del medio.

(ii) La ley de Darcy, que describe la dinámica del flujo a través de un medio poroso, sustituyendo las habituales ecuaciones de Navier-Stokes:

$$\mu\mathbf{V} = -k\nabla p, \tag{0.5}$$

donde μ es la viscosidad del fluido y k la permeabilidad del medio
(iii) La ecuación de estado de los gases perfectos, en el caso de procesos isotermos o adiabáticos:

$$p = p_0 \rho^\gamma, \tag{0.6}$$

donde $\gamma \geq 1$ y p_0 es la presión de referencia.
Combinando estas tres leyes, podemos encontrar la ecuación de la densidad:

$$\rho_t = c\Delta\rho^m, \quad m = 1 + \gamma. \tag{0.7}$$

Del punto de vista matemático, uno puede eliminar la constante c, obteniendo así la PME. En aplicaciones, es por supuesto importante saber con precisión la constante c, que tiene una fórmula exacta en términos de los parámetros del medio. Además, debido al modelo básico, vamos a utilizar con frecuencia terminología tomada de él, como *masa* significando la integral de la solución respecto a la variable espacial, o *presión* para u^{m-1}.

Un modelo comenzando de consideraciones similares ha sido considerado anteriormente por Boussinesq [36] en el estudio de la infiltración del agua en el suelo, y más general en el problema de la filtración de un fluido incompresible a través de un estrato poroso, conduciendo a la PME con el exponente $m = 2$ (también conocida en ingeniería como la ecuación de Boussinesq).

Hay otros importantes modelos a partir de otras áreas diferente de la Mecánica de Fluidos. Uno de éstos modelos es la propagación no lineal del calor en la física del plasma (véase [135]). Otros modelos aparecen en la dinámica de poblaciones, en el límite difusivo de las ecuaciones cinéticas, en la difusión en semiconductores etc. Para el lector interesado, los modelos detallados están presentados en la monografía [131].

Por otra parte, la Ecuación de Difusión Rápida (es decir, la PME con $m < 1$) ha aparecido recientemente de modelos en la física del plasma (la difusión del plasma en varios modelos tiene el coeficiente $A(u) \sim u^{-1/2}$, conduciendo a la PME con $m = 1/2$, [107]). Más recientemente, John King ha obtenido la Difusión Rápida de modelos de difusión de impuridades en silicona, en [92]. Otra aplicación famosa de la difusión rápida es el flujo de Yamabe en la geometría de Riemann. El problema de Yamabe consiste en deformar una métrica de Riemann dada en una métrica con la curvatura escalar constante, en la misma clase conforme. Una derivación geométrica sencilla (ver por ejemplo [127]) conduce a la Ecuación de Difusión Rápida con el exponente $m = (n-2)/(n+2)$.

También recientemente, han sido obtenidos modelos involucrando la llamada difusión super-rápida (PME con $m < 0$, abreviada VFDE). Esta ecuación ha sido propuesta por G. Rosen como un modelo para la conducción no lineal del calor en átomos sólidos de hidrógeno, véase [117], donde se deduce experimentalmente la ecuación con $m = -1$. Después, Chayes, Osher y Ralston han propuesto este modelo para $m < 0$ y $n = 1$ para modelizar avalanchas en pilas de arena, véase [41]. La VFDE en general está usada por Meerson (véase [104]) para describir el enfriamiento de una bola de fuego causada por una fuerte explosión en un gas local.

Para la PLE, el modelo más conocido es probablemente el problema de la filtración de un fluido no newtoniano a través de un medio poroso. Se trata de fluidos donde la relación entre el estrés de cizalladura y la tasa de torcedura es no lineal (mientras para un fluido newtoniano esta relación es lineal y su cociente es precisamente el coeficiente de viscosidad). Así que un coeficiente constante de viscosidad no se puede definir, y las leyes presentadas arriba en la modelización de la PME son más complicadas. Ladyzhenskaia ha estudiado modelos para este tipo de fluidos en [97], llegando a difusiones con dependencia en el gradiente y en particular al flujo p-Laplaciano. Más recientemente, la PLE aparece en modelos de la glaciología o en el estudio de flujos turbulentos a través de medios porosos.

Otro modelo reciente pero muy importante involucrando la PLE y (especialmente) la PLE rápida (es decir, la ecuación en el rango $p < 2$) viene del Procesamiento de Imágenes, en particular en *realce de contorno*, que es un fenómeno de gran interés en aplicaciones de *denoising* y reconocimiento de imágenes, véase por ejemplo [13]. La técnica es considerar una ecuación de evolución para la intensidad de la imagen (vista como una colección de puntos o pixeles), conocida también como *el nivel de gris*, $I(x, y)$, tomando valores en $0 \leq I(x, y) \leq 1$. Como se demuestra en [14] y [13], un modelo realista de realce de imagen involucra las dos condiciones de contorno: $I = 0$ en el lado izquierdo del contorno y $I = 1$ en el lado derecho del contorno. La ecuación de evolución para I ha sido propuesta por Perona y Malik, [111], con la forma general

$$I_t = \operatorname{div}(g(\nabla u)\nabla u), \tag{0.8}$$

donde han sido propuestas varias elecciones para la función g. El modelo original de Perona y Malik tomaba $g(s) = C(1+s)^{-1-\alpha}$, pero hay otros modelos conduciendo a la PLE rápida; por ejemplo, el modelo propuesto por Barenblatt y Vázquez en [14]. De este modelo se obtiene la FPLE con $p < 0$, en dimensión $n = 1$, junto con unas variaciones más generales. Modelos paralelos, conduciendo a problemas singulares de frontera libre que son matemáticamente difíciles pero interesantes (la frontera libre aparece aquí como efecto de la explosión en el gradiente pero no en la propia solución) aparecen en [2].

Finalmente, existen dos ecuaciones límites especiales y muy famosas de la PLE, el caso $p = 1$ y el límite $p \to \infty$. La primera se conoce como la Ecuación de la Variación Total, y aparece en muchos modelos en Procesamiento de Imágenes y Geometría Diferencial. La excelente monografía [3] cubre completamente este asunto. Por otra parte, el proceso límite $p \to \infty$ conduce a la *ecuación del infinito-Laplaciano*, que también es muy interesante. Recomiendo al lector interesado que estudie quizás el primer artículo sobre la evolución gobernada por el infinito-Laplaciano, [86], para entrar en el asunto. No voy a insistir en estos casos porque ellos no entran profundamente en el tema de esta memoria.

Propiedades generales de las PLE y PME

Como ya hemos dicho, la PME y la PLE son los representantes básicos de la clase de ecuaciones de difusión no lineal, pero también, del punto de vista matemático, de las ecuaciones parabólicas degeneradas o singulares. Así que comparten algunas de las propiedades estándar de las ecuaciones parabólicas, pero por otra parte tienen ambas muchas propiedades y fenómenos interesantes que se deben a su degeneración (o singularidad si $m < 1$ o $p < 2$) específica, conduciendo a nuevos aspectos matemáticos muy interesantes. Vamos a discutir sólo las propiedades especiales que diferencian la PME y la PLE de la clase de las ecuaciones uniformemente parabólicas.

(a) **Existencia de soluciones especiales**. Tanto la PME como la PLE tienen soluciones explícitas interesantes, que juegan un papel fundamental en el desarrollo de la teoría cualitativa general. Las primeras soluciones especiales, de forma auto-semejante, han sido descubiertas por Barenblatt [11] y Zeldovich and Kompaneets [134], con una deducción partiendo de la física. Estas soluciones se conocen en el presente como la solución de Barenblatt (o también KPZ en varias fuentes) y tiene la forma explícita

$$B_C(x,t) = t^{-\alpha} F_C\left(|x|t^{-\beta}\right), \quad F(\eta) = \left(C - k\eta^2\right)_+^{\frac{1}{m-1}}, \tag{0.9}$$

donde $C > 0$ es una constante libre y

$$\alpha = \frac{n}{mn-n+2}, \ \beta = \frac{1}{mn-n+2}, \ k = \frac{m-1}{2m(mn-n+2)}. \tag{0.10}$$

Vamos a enfatizar algunas propiedades de estas soluciones especiales. Desde luego, por una simple integración, se ve que la masa de la solución está conservada, en acuerdo con las leyes físicas del fenómeno de difusión. Sea $M = M(C)$ la masa de B_C. Es inmediato

$$\lim_{t\to 0} B_C(x,t) = M\delta(x),$$

diciendo que la solución de Barenblatt es la solución fundamental de la PME en el rango $m > 1$, y se conoce en literatura también como la *solución tipo fuente*, porque modeliza la difusión no lineal del calor saliendo de una fuente puntual. Finalmente, observamos que a cualquier tiempo t, la solución de Barenblatt tiene soporte compacto, y su soporte avanza con velocidad finita.

Existe una familia correspondiente de soluciones fundamentales–o de tipo fuente– auto-semejantes de la PLE, con una forma y propiedades muy parecidas. Éstas soluciones se llaman también soluciones de Barenblatt de la PLE y tienen la forma explícita

$$\overline{B}_C(x,t) = t^{-\overline{\alpha}} F_C\left(|x|t^{-\overline{\beta}}\right), \quad F_C(\overline{\eta}) = \left(C - k\eta^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{0.11}$$

donde $C > 0$ es de nuevo una constante libre y

$$\overline{\alpha} = \frac{n}{np-2n+p}, \ \overline{\beta} = \frac{1}{np-2n+p}, \ k = \frac{p-2}{p}\left(\frac{1}{np-2n+p}\right)^{\frac{1}{p-1}}. \tag{0.12}$$

Al igual que en el caso de la PME, la masa de la solución se conserva, también tiene una distribución delta de Dirac como traza inicial (en este sentido se le llama solución fundamental o tipo fuente) y tiene soporte compacto para todo $t > 0$.

Hay muchas otras soluciones auto-semejantes interesantes, también algunas muy interesantes en el caso de la difusión rápida, presentando nuevos fenómenos (como por ejemplo las *soluciones anómalas*). Pero paramos aquí esta breve discusión, dado que dedicamos el Capítulo 2 precisamente a un estudio detallado de las soluciones auto-semejantes de las dos ecuaciones.

(b) **Velocidad finita de propagación y fronteras libres. Incumplimiento del Principio de Máximo fuerte.** Estas propiedades ya han aparecido en las soluciones de Barenblatt de las dos ecuaciones, pero son de hecho unas propiedades generales de las PME y PLE. A partir de un dato inicial con soporte compacto, las soluciones de la PME y de la PLE quedan con soporte compacto para todo tiempo, y sus soportes avanzan con velocidad finita, en contraste con el famoso resultado para la ecuación del calor, donde una solución no negativa se vuelve automáticamente positiva de inmediato en todo el dominio.

Como una consecuencia inmediata de la velocidad finita de propagación, aparece una *frontera libre* (llamada también *interfaz* o *frontera móvil*) separando la región positiva de la solución de la región donde ésta se anula. La frontera libre se mueve con velocidad finita. Un problema teórico importante relacionado con nuestras ecuaciones es establecer propiedades geométricas y/o de regularidad de las fronteras libres; aunque es un problema muy interesante, sale fuera del alcance de esta memoria. En general, todas las fronteras libre se denotarán con Γ en lo que sigue; más preciso, si Q es el dominio de la solución y

$$\mathcal{P}_u(t) = \{(x,t) \in Q : u(x,t) > 0\}, \tag{0.13}$$

su región de positividad, entonces su frontera libre es

$$\Gamma = \partial\mathcal{P}_u(t) \cap Q.$$

A causa de la degeneración/singularidad, el Principio del Máximo Fuerte (y el subsiguiente Principio de Comparación) no funcionan en su máxima generalidad. Para la PME, se aplica en toda la región regular, pero no en los puntos de la frontera libre; para la PLE, la situación es todavía más dramática, porque el principio del máximo no se aplica tanto en la frontera libre, como en los puntos donde el gradiente de la solución se anula. El fallo del principio del máximo es una de las dificultades que tenemos que afrontar en toda la presente memoria, pero sobre todo en los Capítulos 3 y 4.

Ha habido muchos resultados interesantes en la dirección de sustituir el Principio de Comparación con instrumentos similares que sean verdaderos, como el Principio de Comparación de las Concentraciones ([127, 124]) y el Principio de Comparación de las Intersecciones ([62]).

(c) **Propiedades de reescalamiento.** Tanto la PME como la PLE, para todos los valores de m y p, comparten una propiedad muy importante, que la familia formada por sus soluciones es invariante bajo la acción de un grupo de transformaciones, usualmente denominado *grupo de scaling*. En efecto, si $u(x,t)$ es la solución de la PME (respectivamente PLE), y definimos

$$\tilde{u}(x,t) = Ku(Sx, Tt), \tag{0.14}$$

para parámetros reales K, S, $T > 0$, entonces $\tilde{u}$ es una solución de la PME (respectivamente PLE) si y sólo si $K^{m-1}S^2 = T$ (respectivamente $K^{p-2}S^p = T$). Se obtiene de esta forma un grupo 2-paramétrico de reescalamiento. Se puede reducir a un grupo 1-paramétrico imponiendo ciertas condiciones. Por ejemplo, si imponemos la conservación de masa, obtenemos el grupo 1-paramétrico estándar de transformaciones T_λ dado por

$$T_\lambda(u)(x,t) = u_\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t), \tag{0.15}$$

que deja invariantes las soluciones de Barenblatt y es muy importante en toda la teoría cualitativa. Aquí, α y β son los exponentes de Barenblatt tanto para PME como para PLE. Otros grupos interesantes de scaling aparecerán en los siguientes capítulos.

(d) **El fenómeno de extinción en la difusión rápida.** Otro aspecto general importante es que algunos inesperados fenómenos y comportamientos aparecen en el rango de difusión rápida de ambas ecuaciones. Algunos de los más interesantes son la falta de regularización y la extinción, que vamos a analizar brevemente.

Muchas de las propiedades de la difusión lenta estándar siguen siendo válidas en una parte del rango de difusión rápida, delimitado por los exponentes críticos

$$m_c = \frac{n-2}{n}, \quad p_c = \frac{2n}{n+1},$$

en el caso de la PME y PLE respectivamente. Por supuesto, en estos rangos las soluciones propagan con velocidad infinita, similar a la ecuación del calor, así que no hay soluciones con soporte compacto; pero esta es probablemente la única diferencia importante entre los rangos $m > 1$ y $1 > m > m_c$, respectivamente $p > 2$ y $2 > p > p_c$. Las soluciones especiales tienen las mismas fórmulas, ellas todavía existen y representan el comportamiento asintótico, las soluciones para el problema de Cauchy con dato inicial en L^1 existen para todo tiempo y ellas se regularizan de inmediato después del tiempo inicial.

Novadades significativas en la teoría matemática surgen en el rango $m < m_c$ para la PME y $p < p_c$ para la PLE (por ello este rango se llama usualmente *el rango de difusión muy rápida*, *rango bajo de difusión rápida* o incluso *rango malo de difusión rápida*). Primero, en este rango la solución fundamental deja de existir, como han demostrado Brezis y Friedman en [38]: ellos prueban que no hay soluciones con traza inicial una distribución de Dirac, como las soluciones de Barenblatt en el rango complementario.

Otro fenómeno interesante que ocurre en este rango es la *extinción en tiempo finito* (abreviado EFT); es decir, un dato inicial no trivial u_0 produce una solución $u(t)$ que es no negativa en un interval $0 < t < T$ y se anula en $t = T$, en el sentido de que $u(t) \to 0$ cuando $t \to T$. Cuando la solución alcanza el nivel cero en un primer tiempo T (llamado *tiempo de extinción* en la literatura), por la teoría de regularidad, quedará identicamente cero para todos los tiempos $t > T$. Hay varios ejemplos explícitos de soluciones presentando el fenómeno de extinción. Una de ellas es la siguiente solución de variables separadas:

$$U(x,t;T) = k(m)\left(\frac{T-t}{|x|^2}\right)^{\frac{1}{1-m}}, \quad k(m) = 2\left(n - \frac{2}{1-m}\right)^{\frac{1}{1-m}}. \tag{0.16}$$

Además, las soluciones auto-semejantes llamadas de *tipo II*, teniendo la forma

$$u(x,t) = (T-t)^{\alpha} f(x(T-t)^{\beta})$$

se extinguen en tiempo finito si $\alpha > 0$.

El fenómeno de extinción está analizado y explicado en detalle en la reciente monografía por J. L. Vázquez, [127]. Ahí se prueba que en efecto las soluciones auto-semejantes de tipo II son esencialmente las soluciones típicas que representan el proceso de evolución en este rango (jugando el papel de la familia de Barenblatt para $m > m_c$ o $p > p_c$), y que EFT es un fenómeno típico en ese rango. Hay también tasas precisas de extinción de las soluciones y propiedades del tiempo de extinción T, como por ejemplo su dependencia continua del dato inicial.

Otro fenómeno que caracteriza el rango de la difusión muy rápida es la posible falta de regularidad de las soluciones. En efecto, para $m < m_c$ en la PME y $p < p_c$ en la PLE, el efecto regularizante $L^1 - L^\infty$ no se aplica; eso es, soluciones con el dato inicial $u_0 \in L^1$ no necesariamente están acotadas en cualquier tiempo $t > 0$. Para obtener la inmediata acotación, tenemos que pedir un dato inicial u_0 en un espacio funcional mejor, L^r, $r > r_c$, para un cierto exponente crítico r_c (valores precisos se dan en el Capítulo 4). Además, un fallo de regularidad todavía más fuerte ocurre, conocido como *the backward effect*: si el dato inicial $u_0 \in L^r$ con $r < r_c$, entonces la solución es sólo L^1 para tiempos positivos.

Finalmente, en artículos recientes, se estudia también el rango aún más singular $m \leq 0$ en la PME y $p < 1$ en la PLE; eso se llama generalmente *rango de difusión muy rápida* o *rango de difusión super-rápida*. En estos casos, los fenómenos matemáticos son aún más sorprendentes, como por ejemplo la no existencia de soluciones con dato inicial integrable (en L^1).

En el presente trabajo, nos vamos a encontrar con estos interesantes fenómenos en los Capítulos 2 y 4, que son en una parte esencial dedicados a la difusión rápida. Para una presentación completa de dichos fenómenos, recomiendo el libro reciente [127].

Estructura y descripción de la memoria

Esta memoria contiene cuatro partes diferentes, tratando cuatro problemas relacionados con las dos ecuaciones de difusión no lineal básicas, con énfasis especial en la PLE. En los cuatro capítulos principales, estudiamos varios problemas típicos relacionados con esta clase de ecuaciones: soluciones especiales, comportamiento asintótico, regularidad, desigualdades de Harnack y estudio del comportamiento de la frontera libre. Se estudia también la influencia de un extra-término, de la forma de una potencia del gradiente, en el caso más difícil, del exponente crítico (resonante). Vamos a presentar ahora de manera más precisa los asuntos tratados y los resultados obtenidos en esta memoria.

(a) En el **Capítulo 2** se estudian las soluciones con simetría radial y, en particular, auto-semejantes, de las dos ecuaciones. La importancia de estas soluciones especiales para la teoría cualitativa es fundamental y discutida en la sección anterior.

El resultado principal del Capítulo 2 es claramente la relación explícita, sencilla pero muy interesante, que transforma las soluciones radiales de la PME en soluciones radiales de la PLE y recíproco. Esta relación es muy simple en dimensión $n = 1$, donde viene dada por integración directa con respecto a la variable espacial. Además, en la literatura ha habido muchas observaciones sobre propiedades similares de la PME y de la PLE en varias dimensiones, y esto nos motivo en la búsqueda de una fórmula general explícita explicando éstas similaridades.

Nuestras relaciones generalizan la integración que ya se conocía en dimension $n = 1$. Se obtiene de manera esencial también via integración, pero en varias dimensiones involucra un cambio de dimensión entre la PME y la PLE, y la multiplicación (antes de integrar) por una potencia de la variable radial independiente.

En el resto del capítulo, se aplican éstas transformaciones y algunas técnicas de Sistemas Dinámicos (principalmente *análisis de planos de fases*) para estudiar y clasificar las soluciones auto-semejantes de las dos ecuaciones, es decir, las soluciones de la PME y de la PLE teniendo una de las siguientes tres formas:

$$\left\{\begin{array}{l} u(x,t) = t^{-\alpha} f(xt^{-\beta}), \\ u(x,t) = (T-t)^{\alpha} f(x(T-t)^{\beta}), \\ u(x,t) = e^{-\alpha t} f(xe^{-\beta t}), \end{array}\right. \tag{0.17}$$

donde f es el perfil de la solución; éstas soluciones se llaman soluciones auto-semejantes de tipo I, II y III respectivamente.

En particular, obtenemos nuevas soluciones auto-semejantes interesantes de la PLE que no se conocían antes, como la solución tipo dipolo de la PLE (que va a ser usada en el Capítulo 3 para estudiar el comportamiento asintótico en dominios con agujeros) y las soluciones anómalas para la PLE. También encontramos una *sucesión tipo Hulshof* de exponentes dando lugar a soluciones auto-semejantes con soporte compacto de la PLE, correspondientes a la conocida serie obtenida por Hulshof en [75] para la PME. La última parte del capítulo está dedicada a un estudio completo de las soluciones auto-semejantes en el rango de difusión super-rápida ($m < 0$ para la PME and $p < 1$ para la PLE), que es interesante pero muy poco explorado hasta ahora.

Los resultados del Capítulo 2 corresponden esencialmente a los publicados en los artículos [78] y [79].

(b) En el **Capítulo 3**, estudiamos el comportamiento asintótico de las soluciones del problema de Dirichlet homogeneo para la PLE en un dominio con agujeros. Este estudio tiene como punto de partida los resultados similares obtenidos por C. Brändle, F. Quirós y J. L. Vázquez en [37] para la PME en un dominio con agujeros. Más precisamente, estudiamos el siguiente problema:

$$\left\{\begin{array}{l} u_t = \Delta_p u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{array}\right. \tag{0.18}$$

donde $p > 2$ y $\Omega = \mathbb{R}^n \setminus G$ es el complemento de un conjunto abierto acotado. Damos condiciones precisas sobre el dato al inicio del capítulo correspondiente.

Reduciendo el problema al caso más sencillo de un dominio exterior, y siguiendo las ideas generales del estudio hecho para la PME, el análisis de este problema se divide en tres casos, con respecto a la relación entre la dimensión n y el exponente p. Tenemos así:

(i) Para $n > p$, probamos que los agujeros no juegan ningún papel esencial en el comportamiento asintótico. Este es el caso más fácil, y el hecho técnico importante aquí es que, si reducimos el agujero al origen por el proceso de reescalamiento estándar, al final la singularidad puntual obtenida es eliminable. Esto permite usar técnicas generales válidas en todo el espacio, esencialmente el conocido Método de los Cuatro Pasos (véase [128]). Así, el comportamiento asintótico está dado en la región exterior (lejos del agujero) por un representante de la familia de soluciones Barenblatt (1.11); el perfil límite es único cuando el dato inicial u_0 está dado, la unicidad se prueba analizando la masa del perfil límite.

(ii) Para $n = p$, estamos en el caso crítico (resonante). Como es habitual en estos casos, se espera que aparezcan ciertas correcciones de orden logarítmico tanto en la tasa de decaimiento como en la tasa de expansión del soporte. Para adivinarlas, primero se adapta un cálculo formal de Gilding y Gonzerkiewicz [69] conduciendo a las tasas exactas. Se obtiene que el perfil asintótico está dado en este caso no por una solución de la PLE o por la parte estacionaria de alguna versión reescalada de ella, sino por una subsolución de la PLE, que se parece a una solución de Barenblatt, pero con algunas tasas logarítmicas; más preciso, el perfil límite es

$$U(x,t;C) = (t\log\, t)^{-\frac{1}{p-1}} \left(C - k|x|^{\frac{p}{p-1}}(\log\, t^{\frac{p-2}{(p-1)^2}}\right)_+^{\frac{p-1}{p-2}}.$$

Para probar de forma rigurosa la convergencia asintótica a un (único cuando el dato inicial está dado) perfil $U(x,t;C)$, necesitamos técnicas más poderosas, porque la de los cuatro pasos no funciona aquí. En cambio, podemos aplicar en este problema la técnica de estabilidad desarrollada por Galaktionov y Vázquez en [62]. Para utilizarla, necesitamos primero un paso de *scaling continuo*, tomando en cuenta la escala logarítmica exacta introducida por la resonancia y transformando la ecuación en una ecuación tipo Fokker-Planck no lineal con p-Laplaciano y una perturbación asintóticamente pequeña. Observamos aquí (como en muchos otros problemas en EDP y en Matemática Aplicada en general) como el entendimiento de la corrección logarítmica y su correcta adivinanza son esenciales para aplicar correctamente la técnica rigurosa y probar el resultado ya esperado después de los cálculos formales.

(iii) El caso de baja dimensión, $n < p$, es claramente el más interesante, difícil e inesperado de los tres. Debido a la dimensión pequeña, el agujero tiene una influencia dramática, y la singularidad que crea en el origen en el límite después del rescaling es ahora esencial. Así que es natural pensar en un perfil auto-semejante de la PLE teniendo una singularidad en $x = 0$.

Teniendo ya en la mente el estudio completo de las soluciones auto-semejantes efectuado en el Capítulo 2, descubrimos que una familia de perfiles candidatos existe exactamente en dimensión $n < p$ y es la familia llamada *tipo dipolo* de la PLE. Esta es una familia completamente nueva de perfiles de la PLE, descubierta por nuestro análisis en el Capítulo 2, que tiene la forma:

$$D_\lambda(x,t) = t^{-\alpha}F(xt^{-\beta}), \quad F_\lambda(\eta) = \lambda^p F(\lambda^{2-p}\eta), \quad \forall\, \lambda > 0, \tag{0.19}$$

donde los exponentes auto-semejantes satisfacen la relación

$$(p-2)\alpha + p\beta = 1, \quad \alpha > 0, \quad \beta > 0, \tag{0.20}$$

pero ambos exponentes y el perfil F_λ no tienen fórmulas explícitas. Además, estos perfiles se llaman *anómalos* porque no se obtienen por una ley de conservación, pero como una órbita especial en el plano de fases asociado a una ODE. Vamos a detallar estos hechos en el Capítulo 3. También obtenemos que $F_\lambda(0) = 0$, pero su derivada es singular en $\eta = 0$. Más preciso, cerca de $\eta = 0$ tenemos:

$$F(\eta) \sim \eta^{(p-N)/(p-1)}, \quad \text{cuando} \quad \eta \sim 0. \tag{0.21}$$

Demostramos que, en efecto, esta familia de soluciones auto-semejantes anómalas de la PLE da el comportamiento asintótico en un dominio exterior en dimensión $n < p$. En este caso se usa una técnica diferente en la demostración, la de considerar barreras óptimas, con precedentes en la literatura en artículos como [61, 88]. En nuestro caso, su aplicación es más complicada debido a la no existencia de una fórmula explícita del perfil límite esperado y debido a la falta del Principio de Máximo Fuerte. En efecto, la demostración consiste en una serie de observaciones geométricas y topológicas, y un muy delicado *análisis de contacto*, cuya idea es de eliminar los posibles contactos entre el perfil límite y la barrera óptima por abajo, en caso de no coincidencia entre ellos. Este análisis de contacto, basado en la existencia de desigualdades de Harnack y de ciertas barreras intermedias, es el complicado argumento que sustituye el principio del máximo en los puntos degenerados de una solución tipo dipolo.

El material del Capítulo 3 está esencialmente publicado en los artículos [80] y [81].

(c) En el **Capítulo 4**, realizamos un estudio más teórico de la PLE en su rango de difusión rápida $1 < p < 2$, con énfasis en la regularidad de las soluciones y en la expansión de la positividad desde el tiempo inicial a tiempos ulteriores.

Se sabe, de los trabajos de DiBenedetto, Gianazza, Vespri, Urbano etc. (véase por ejemplo [57]) que la PLE estándar tiene buenas propiedades regularizantes en el rango $p > 2$. Más preciso, en dicho rango cualquier solución débil local con dato inicial $u_0 \in L^1_{loc}(\Omega)$ para un dominio $\Omega \in \mathbb{R}^n$, se regulariza de inmediato, en el sentido de que $u \in C^{1,\alpha}$ en todo tiempo $t > 0$. De hecho, esta prueba muy complicada se realiza en dos pasos: en el primer paso, un resultado general de regularidad dice que las soluciones que son localmente acotadas son continuas Hölder; luego, en un segundo paso, se prueba que una solución con dato inicial $u_0 \in L^1_{loc}$ es automáticamente localmente acotada. Además, se dan estimaciones dando acotaciones de la norma local L^∞ en términos de las normas locales L^p, estimaciones llamadas *efectos regularizantes locales*, y se calculan las constantes óptimas. Luego, se dan acotaciones por abajo, representando la persistencia y la expansión de la positividad cuando se empieza con un dato inicial soportado en una bola, y ambas estimaciones se pueden combinar para obtener varias formas de desigualdades de Harnack (véase [57]).

En todo caso, las cosas se vuelven más complicadas cuando $p < 2$ y especialmente cuando p se acerca a 1. Un punto crítico esencial es $p_c = 2n/(n+1)$. En el rango de difusión rápida $p < 2$ y en especial para $p < p_c$, las estimaciones conocidas como efectos regularizantes locales no son óptimas, tanto cualitativamente como cuantitativamente; además, una estimacion óptima de la expansión de la positividad, conduciendo a una acotación por abajo de cualquier solución débil local u, faltaba en este rango, y eso fue un problema abierto propuesto por DiBenedetto y sus colaboradores, de encontrar formas adecuadas de las desigualdades de Harnack para $p < p_c$ (véase [55, 54]). En el Capítulo 4, analizamos este problema y proponemos una solución.

A. *Efectos regularizantes locales óptimos.* En la primera parte, establecemos nuevos efectos regularizantes óptimos, probando, como ya era conocido en el caso de la PME, que la regularización, para $p < p_c$, se cumple si empezamos con un dato inicial no en L^1_{loc}, sino en un espacio mejor L^r_{loc}, donde $r > r_c = n(2-p)/p$. Obtenemos también estimaciones locales óptimas de la norma L^∞ de la u en términos de las normas locales L^r, con constantes óptimas exactas.

B. *Soluciones grandes.* Probamos la existencia de soluciones grandes en un dominio acotado, es decir soluciones que están acotadas en el interior del dominio y convergen a $+\infty$ al acercarse a la frontera del dominio. Obtenemos la tasa asintótica exacta de la convergencia a $+\infty$ cerca de la frontera. Este resultado es nuevo y, aparte del interés en si mismo, produce una clase muy importante de soluciones para aplicaciones ulteriores. De hecho, en todo el Capítulo 4, usamos las soluciones grandes como funciones de comparación.

C. *La persistencia de la positividad.* Se trata de estimaciones inferiores de la solución local u en términos de algunas normas locales de su traza inicial, teniendo la forma general

$$\inf_{x\in B_R(x_0)} u^{p-1}(x,t) \geq CR^{p-n}t^{\frac{p-1}{2-p}} \int_{B_R(x_0)} u_0(x)\ \mathrm{d}x,$$

para todo $0 < t < t^*$. En particular, estas desigualdades juegan el papel de una Harnack inferior y se pueden usar para estimar, si empezamos con una cierta cantidad de un fluido concentrado en una bola pequeña, por cuanto tiempo y en que cantidades el fluido se queda en la bola inicial. Esta aplicación justifica el nombre. Obtenemos también una fórmula precisa para el tiempo crítico t^* y fórmulas mejoradas, con constantes exactas y dependencia óptima, en particular para el rango $p < p_c$, que es el más interesante en esta línea de investigación.

D. *Desigualdades de Harnack intrínsecas.* Finalmente, sólo juntando las acotaciones superiores óptimas (efectos regularizantes) obtenidas en la parte A con las acotaciones inferiores en la parte C, establecemos varias formas de desigualdades de Harnack generalizadas. En estas formas, como es habitual en el caso de la difusión rápida, las constantes no son globales, ellas dependen usualmente en cantidades relacionadas con la misma función. Para llegar a una forma más interesante y útil, tenemos que aceptar que los cilindros parabólicos donde la desigualdad se cumple dependan de su centro. Ésta es la geometría intrínseca asociada a la ecuación, y de esta manera podemos obtener desigualdades de Harnack en forma clásica. Vamos a hacer preciso este asunto en el Capítulo 4.

E. *La Desigualdad Especial de Energía.* Se trata de una desigualdad muy corta, natural e interesante que deducimos y cuyas aplicaciones exploramos. Si u es una solución débil local continua de la ecuación p-Laplaciana en un cilindro $Q_T = \Omega \times (0,T)$, con $1 < p < 2$, y $0 \leq \varphi \in C^2_c(\Omega)$ es cualquier función test admisible, entonces $u_t = \Delta_p u \in L^2_{loc}(Q_T)$ y la siguiente desigualdad se cumple:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p\varphi\ \mathrm{d}x + \frac{p}{n}\int_\Omega (\Delta_p u)^2\varphi\ \mathrm{d}x \leq \frac{p}{2}\int_\Omega |\nabla u|^{2(p-1)}\Delta\varphi\ \mathrm{d}x, \tag{0.22}$$

en sentido distribucional en $\mathcal{D}'(0,T)$. Se deducen y estudian también varias de sus aplicaciones.

Los resultados de este capítulo han sido obtenidos en colaboración con Matteo Bonforte y publicados en el artículo [33].

(d) En el **Capítulo 5**, se estudia el comportamiento asintótico y la expansión de la frontera libre para la PLE con absorción en el gradiente. Más preciso, estudiamos el comportamiento asintótico para el siguiente problema de Cauchy en $Q = \mathbb{R}^n \times (0, \infty)$:

$$\begin{cases} \partial_t u - \Delta_p u + |\nabla u|^q = 0, & (x,t) \in Q, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{0.23}$$

con dato inicial $u_0 \geq 0$, soporte compacto, y $p > 2$. El interés matemático del problema (aparte del interés en aplicaciones) es estudiar la influencia del término en gradiente sobre la evolución; en general, la presencia de términos en gradiente puede complicar mucho la evolución y, equilibrando los efectos de la difusión y del término en gradiente, se pueden obtener resultados interesantes y poco esperados.

En este caso, el efecto del término en gradiente depende, por supuesto, del exponente q. Para q muy grande, se espera que el término de difusión domine en la evolución; por el contrario, para q muy pequeña, el término en gradiente es más fuerte y se espera que domine. Eso sugiere también la existencia de exponentes críticos intermedios q donde cosas poco usuales pueden pasar.

En efecto, trabajos anteriores sobre la regularidad de las soluciones y estimaciones del gradiente para tanto el caso lineal $p = 2$ como el caso no lineal $p > 2$ (véase [20, 64, 18]) han identificado dos exponentes críticos:

$$q_1 = p - 1, \quad q_2 = p - \frac{n}{n+1}, \tag{0.24}$$

que delimitan aspectos diferentes en el comportamiento asintótico. En el caso $1 < q < q_1$, el problema ha sido resuelto por Laurençot y Vázquez en [99], donde han demostrado que el término de difusión es despreciable en el límite asintótico y la absorción en el gradiente domina. El perfil asintótico es una *pila de arena* fija, obtenida como solución de la ecuación Hamilton-Jacobi tipo Eikonal que queda después de eliminar la difusión.

El resultado original de este capítulo se refiere al mismo problema del comportamiento asintótico y de la expansión de la interfase en el caso crítico $q = p - 1$, donde técnicas similares a las del artículo [99] no funcionan. Este caso ha determinado un análisis delicado y complicado, tratando de equilibrar, como es usual en casos resonantes, los efectos de los dos procesos compitiendo en la ecuación.

En este caso, el perfil límite tiene una forma complicada, siendo en una escala simplificada de nuevo una pila de arena (no regular en el origen) con una cúspide en el vertice. Se obtiene como la solución de una ecuación estacionaria tipo Hamilton-Jacobi, pero después de varios pasos de scaling. La técnica de la prueba es complicada y está basada en dos pasos sucesivos de scaling y una construcción técnicamente complicada de subsoluciones teniendo como punto de partida un análisis de las ondas viajeras de la ecuación. Es una prueba que mezcla muchos elementos de matemáticas difíciles. Además, la comparación con las subsoluciones tipo ondas viajeras y un primer paso de scaling (pasando al *tiempo logarítmico*) permiten establecer el comportamiento de la frontera libre, que, como se esperaba, tiene una tasa logarítmica de avance.

Los resultados de este capítulo han sido tomados del artículo [77].

Chapter 1

Introduction

Among all the fields of Mathematics, the field of Partial Differential Equations is one of those that describe better the models coming from applied sciences as Physics, Chemistry, Engineering, Image Processing and Biology. Moreover, due to the large number of applications, but also to the big complexity of their mathematical theory, the Partial Differential Equations have motivated and stimulated the study of other areas in Mathematics, that today are very important and widely studied, like Harmonic Analysis, Numerical Analysis, Abstract Functional Analysis and Operator Theory.

1.1 Nonlinear Diffusion Equations. General features

In this memoir we deal with a particular type of Partial Differential Equations, that are known (due to their physical relevance, as described below) as Nonlinear Diffusion Equations. This class of equations has as prototypes the following two important models:

$$u_t = \Delta u^m, \tag{1.1}$$

known as the Porous Medium Equation (PME for short), and

$$u_t = \Delta_p u, \tag{1.2}$$

known as the p-Laplacian evolution equation (PLE for short), where

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

These equations are two of the simplest representatives in the class of nonlinear evolution equations of parabolic type. They arise from many phenomena from applied sciences, that motivated, together with their richness in unusual mathematical properties, their extensive study in the last decades. Moreover, although formally they can be seen as nonlinear variations of the classical Heat Equation (HE for short),

$$u_t = \Delta u, \tag{1.3}$$

that can be recovered from the PME with $m = 1$ and from the PLE with $p = 2$, their mathematical properties depart strongly from those of the HE. More than that, different exponents m and p give rise to very different properties of the solutions; in particular, there exist several critical exponents where the behavior of the solutions changes in an essential way.

Both models we deal with, namely the PME and the PLE, are nonlinear evolution equations, formally of parabolic type. If we write them in the standard divergence form,

$$u_t = \mathrm{div}(A(u, Du)\nabla u),$$

we remark that the diffusion coefficient $A(u, Du)$ is $A(u) = mu^{m-1}$ (under the restriction $u \geq 0$) for the PME, and $A(u) = |\nabla u|^{p-2}$ for the PLE. We thus remark that both equations could be degenerate or singular. More precisely, the PME with $m > 1$ is strictly parabolic only where $u \neq 0$, and at the points where u vanishes, the equation degenerates for $m > 1$ (since $A(u) = 0$ in this case) or becomes singular for $m < 1$ (since $A(u) = \infty$ in this case). The same is true for the PLE, but here the influent term is the modulus of the gradient (we say that the PLE is the prototype of Gradient-Dependent Diffusion Equations). At the points where $\nabla u = 0$, the PLE degenerates if $p > 2$ and becomes singular if $p < 2$. That is why, since the historically more important and more studied range of exponents is $m > 1$, respectively $p > 2$, these equations are called *degenerate parabolic.*

The change between the two ranges $m > 1$ and $m < 1$ in the PME (the latter being known in literature as the *Fast Diffusion Equation*, FDE for short), and between the ranges $p > 2$ and $p < 2$ (the latter being called in the sequel the *Fast p-Laplacian Equation*, by similarity with the Fast Diffusion) reflects strongly on the basic properties of the solutions. We will make this precise below in the subsection about special features of the two equations.

1.1.1 Models and applications

Both PME and PLE arise in the description of different natural phenomena. We will briefly describe some of their more important or well-known practical applications.

For the standard PME ($m > 1$), maybe the best known application (which justifies the name given to the equation) is the description of the flow of an isentropic gas through a porous medium. This has been modeled independently by Muskat [106] and Leibenzon [101], with a deduction based on Darcy's law. This is a very well-known model, but since it may be considered as the starting point for the development of the theory, I will sketch it here for the reader's convenience. The model can be formulated in terms of the variables density (denoted by ρ), pressure (denoted by p) and velocity field (denoted by $\mathbf{V}$), which are related by the following three physical laws:

(i) Continuity equation in fluid mechanics:

$$\varepsilon\rho_t + \mathrm{div}(\rho\mathbf{V}) = 0, \tag{1.4}$$

where $\varepsilon \in (0, 1)$ is the porosity of the medium.

(ii) Darcy's law, which describes the dynamics of the flow through porous media, replacing the usual Navier-Stokes equations:

$$\mu \mathbf{V} = -k\nabla p, \tag{1.5}$$

where μ is the viscosity of the fluid and k the permeability of the medium.
(iii) State equation for perfect gases, in the case of isothermal or adiabatic processes:

$$p = p_0 \rho^{\gamma}, \tag{1.6}$$

where $\gamma \geq 1$ and p_0 is the reference pressure.

By combining these three laws, we can find the equation satisfied by the density:

$$\rho_t = c\Delta\rho^m, \quad m = 1 + \gamma. \tag{1.7}$$

From the mathematical point of view, one can scale out the constant c, thus arriving to the PME. In applications, of course it is important to know the value of the constant c, but this has a precise form that can be calculated in terms of the parameters of the medium. Moreover, due to this basic model, we will often use in the sequel terminology taken from it, such as *mass* for the space integral of the solution, and *pressure* for the quantity u^{m-1}.

A model starting from similar considerations has been considered previously by Boussinesq [36] in the study of groundwater infiltration, and more generally in the problem of filtration of an incompressible fluid through a porous stratum, leading to the PME with the particular exponent $m = 2$ (also known in engineering as Boussinesq equation).

There are also important models coming from other areas than Fluid Mechanics. One of these is the nonlinear heat propagation with thermal dependent conductivity, whose main applications are in plasma physics (see [135]). Other models appear in population dynamics, in diffusive limit of kinetic equations, in diffusion in semiconductors etc. For the interested reader, the detailed models are presented in the monograph [131].

On the other hand, the Fast-Diffusion Equation (that is, the PME with $m < 1$) appeared recently in models from plasma physics (the plasma diffusion in several models has the diffusion $A(u) \sim u^{-1/2}$, leading to the PME with $m = 1/2$, [107]). More recently, John King obtained the Fast Diffusion Equation from models of diffusion of impurities in silicon, in [92]. Another celebrated application of the Fast Diffusion is the Yamabe flow in Riemannian geometry. The Yamabe problem is to deform a given Riemannian metric into a metric of constant scalar curvature, in the same given conformal class. A simple geometric derivation (see for example [127]) leads to the Fast Diffusion Equation with the precise exponent $m = (n-2)/(n+2)$.

Also recently, models leading to the so-called Superfast diffusion ($m < 0$) were found. This equation was proposed by G. Rosen as a model for the nonlinear heat conduction in solid hydrogen atoms, see [117], where the equation for $m = -1$ is deduced experimentally. Later on, Chayes, Osher and Ralston have proposed this model for $m < 0$ and $n = 1$ to model avalanches in sandpiles, see [41]. The VFDE in general is also used by Meerson (see [104]) to describe the cooling of a fireball caused by a strong explosion of a local gas.

For the PLE, the best known model is probably the problem of the filtration through a porous medium for a Non-Newtonian fluid. These are fluids where the relation between the shear stress and the strain rate is nonlinear (while for a Newtonian fluid this relation is linear and their quotient is precisely the viscosity coefficient). Therefore a constant coefficient of viscosity cannot be defined, and the laws described in the modeling of the PME above are more complicated. Ladyzhenskaia studied models for this kind of fluids in [97], arriving to gradient-dependent diffusions and in particular to the p-Laplacian flow. More recently, the PLE arose in models from Glaciology or in the study of turbulent flows through porous media.

Another recent but very important model involving the PLE and (mainly) the Fast PLE (that is, the equation in the range $p < 2$) comes from Image Processing, particularly in *contour enhancement*, which is a phenomena of great interest in applications of denoising and recognition of images, see for example [13]. The technique is to consider an evolution equation for the intensity of the image (seen as a collection of points or pixels), also known as the *grey level*, $I(x, y)$, taking values in $0 \leq I(x, y) \leq 1$. As showed in [14] and [13], a realistic model of image enhancement involves the two contour-conditions: that $I = 0$ on the left-hand side of the contour and $I = 1$ on the right-hand side of the contour. The evolution equation for I has been proposed by Perona and Malik, [111], having the general form

$$I_t = \mathrm{div}(g(\nabla u)\nabla u), \tag{1.8}$$

where different choices of g has been proposed. The original model by Perona and Malik took $g(s) = C(1+s)^{-1-\alpha}$, but there are other models leading to the Fast PLE; for example, the one proposed by Barenblatt and Vázquez in [14]. From this model the FPLE with $p < 0$ arises, in dimension $n = 1$, and some other more general variants of it. Some parallel models, leading to mathematically difficult but interesting singular free boundary problems (where the free boundary appears as the effect of the blow-up in the gradient, but not of the solution itself) appear in [2].

Finally, there exists two special and famous limit equations of the PLE, the case $p = 1$ and the limit $p \to \infty$. The former is well-known as the Total Variation Flow, and appears from many models in Image Processing and in Differential Geometry. The excellent monograph [3] covers completely the subject. On the other hand, the limiting process $p \to \infty$ leads to the *infinity-Laplacian equation*, which is also of great interest. I recommend to the interested reader to study maybe the first paper on the evolution governed by the infinity-Laplacian, [86] to enter the subject. I will not insist on these cases since they do not enter deeply in the framework of this memoir.

1.1.2 General features of the PLE and PME

As we have already said, the PME and the PLE are the basic representatives of the class of Nonlinear Diffusion Equations, but also, from the mathematical point of view, of the class of Degenerate or Singular Parabolic Equations. Hence, they will share some of the standard properties of parabolic equations, but on the other hand they will both present many interesting properties and phenomena which are due to their specific degeneracy (or singularity if $m < 1$ or $p < 2$), this fact giving rise to new and very interesting mathematical features. We will only discuss the special features which difference PME and PLE from the class of uniformly parabolic equations.

(a) **Existence of special solutions**. Both PME and PLE provide interesting explicit, particular solutions that play a fundamental role in the development of the general qualitative theory. The first special solution, in self-similar form, was found by Barenblatt [11] and Zeldovich and Kompaneets [134], with a deduction starting from physics. It is now known generally as the Barenblatt solution (also named in some texts ZKB solution) and has the explicit form

$$B_C(x,t) = t^{-\alpha} F_C\left(|x|t^{-\beta}\right), \quad F(\eta) = \left(C - k\eta^2\right)_+^{\frac{1}{m-1}}, \tag{1.9}$$

where $C > 0$ is a free constant and

$$\alpha = \frac{n}{mn-n+2}, \ \beta = \frac{1}{mn-n+2}, \ k = \frac{m-1}{2m(mn-n+2)}. \tag{1.10}$$

Let us emphasize some features of this special solution. First of all, by a simple integration we remark that the mass of the solution is conserved, in accordance with the physical laws of the diffusion phenomenon. Let $M = M(C)$ be the mass of B_C. Then, it is immediate that

$$\lim_{t\to 0} B_C(x,t) = M\delta(x),$$

hence the Barenblatt solution is the fundamental solution of the PME in the range $m > 1$, and it is also called in literature the *source-type solution*, since it models the nonlinear diffusion of heat starting from a point source. Finally, we remark that at any time t, the Barenblatt solution is compactly supported, and its support advances with finite speed.

There exists a corresponding family of self-similar fundamental–or source type–solutions for the PLE, having a very similar form and similar features. These solutions will be also called Barenblatt solutions of the PLE and have the explicit form:

$$\overline{B}_C(x,t) = t^{-\overline{\alpha}} F_C\left(|x|t^{-\overline{\beta}}\right), \quad F_C(\overline{\eta}) = \left(C - k\eta^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{1.11}$$

where $C > 0$ is again a free constant and

$$\overline{\alpha} = \frac{n}{np-2n+p}, \ \overline{\beta} = \frac{1}{np-2n+p}, \ k = \frac{p-2}{p}\left(\frac{1}{np-2n+p}\right)^{\frac{1}{p-1}}. \tag{1.12}$$

As in the case of the PME, the mass of the solution is conserved, then it has initial trace a Dirac delta (in this sense it is called fundamental or source-type solution) and it also has compact support at any time $t > 0$.

There are many other interesting self-similar solutions, also some very interesting solutions in the Fast Diffusion range ($m < 1$ and $p < 2$) providing new phenomena (for example the *anomalous solutions*). But we stop here with this short presentation, since we dedicate Chapter 2 precisely to a detailed study of the self-similar solutions of the two equations.

(b) **Finite speed of propagation and free boundaries. Failure of the Strong Maximum Principle.** This has already appeared in the features of the Barenblatt solutions of both equations, but it is in fact a general property of the PME and PLE. Starting from a compactly supported initial data, the solutions of the PME and of the PLE remain compactly supported for all times, and their support advances with finite speed, contrasting with the celebrated result for the Heat Equation, where a nonnegative solution of it is automatically positive everywhere in its domain.

As an immediate consequence of the finite speed of propagation, a *free boundary* (also called *interface* or *moving boundary*) separating the positive region of the solution and the zero region appears. This free boundary moves with finite speed. An important theoretical problem related to our type of equations is to establish geometric properties and/or regularity of the free boundary; although it is a very interesting problem, it is outside the scope of the present memoir. Generally, all free boundaries will be denoted by Γ in the sequel; more precisely, if we denote Q the domain of the solution and

$$\mathcal{P}_u(t) = \{(x,t) \in Q : u(x,t) > 0\}, \tag{1.13}$$

the positivity region, then the free boundary is

$$\Gamma = \partial \mathcal{P}_u(t) \cap Q.$$

Because of the degeneracy/singularity, the Strong Maximum Principle (and the subsequent Comparison Principle) fails to hold in its whole generality. For the PME, it holds in any regular point, but not on the free boundary points; for the PLE, the situation is even more dramatic, since the Strong Maximum Principle fails to hold both on the free boundary, but also at points where the gradient of the solution vanishes. The failure of the Strong Maximum Principle is one of the difficulties that we have to face in all the present work, but specially in Chapters 3 and 4.

There are many interesting results in the direction of substituting the Comparison Principle with similar tools that hold true, such as the Concentration Comparison Principle ([127, 124]) and the Intersection Comparison ([62]).

(c) **Scaling properties.** Both the PME and the PLE, for all values of m and p, share a very important property, that the family of their solutions is invariant under a group of transformations, usually called the *scaling group*. Indeed, if $u(x,t)$ is a solution of the PME (respectively PLE), and we define

$$\tilde{u}(x,t) = Ku(Sx, Tt), \tag{1.14}$$

for real parameters K, S, $T > 0$, then $\tilde{u}$ is a solution of the PME (respectively of the PLE) if $K^{m-1}S^2 = T$ (respectively $K^{p-2}S^p = T$). Hence, we obtain a 2-parameter scaling group. It can be reduced to a 1-parameter group by imposing some extra condition. For example, if we impose the conservation of mass, we obtain the standard 1-parameter scaling group T_λ, given by

$$T_\lambda(u)(x,t) = u_\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t), \tag{1.15}$$

which leaves invariant the Barenblatt solutions and is very important in the whole theory. Here, α and β are the Barenblatt exponents, both for the PME and for the PLE. Other interesting scaling groups will appear throughout the Chapters of the thesis.

(d) **Extinction phenomenon in the Fast Diffusion case.** Another important general aspect is that some unexpected mathematical phenomena and behavior appear in the Fast Diffusion range of both equations. Some of the most interesting are the lack of regularization and extinction, that we briefly discuss below.

Many of the properties and features of the slow-diffusion case remain still valid in a part of the Fast Diffusion range, delimited by the critical exponent

$$m_c = \frac{n-2}{n}, \quad p_c = \frac{2n}{n+1},$$

in the case of PME and PLE respectively. Of course, in this range solutions propagate with infinite speed, similarly as for the Heat Equation, thus there is no compactly supported solution; but this is maybe the only important difference. The special solutions have the same formulas, they still exist and represent the asymptotic behavior, solutions of the Cauchy problem with L^1 initial data exist for all time and they regularize immediately after the initial time.

Significant novelties in the mathematical theory appear in the range $m < m_c$ for the PME and $p < p_c$ for the PLE (that is why this range is called usually the *very fast diffusion range*, the *lower fast diffusion range* or even the *bad fast diffusion range*). First of all, in this range the fundamental solution ceases to exist, as Brezis and Friedman have proved in [38]: they show that there is no solution with initial trace a Dirac mass, as there was the Barenblatt family in the slow diffusion range.

Moreover, one interesting phenomenon that occurs in this range is the *extinction in finite time* (for short EFT in the sequel); that is, some nontrivial initial data u_0 produce a solution $u(t)$ that is nonnegative in some interval $0 < t < T$ and vanishes at time $t = T$, in the sense that $u(t) \to 0$ as $t \to T$. Once the solution touches the level zero at some first time T (called *extinction time* in literature), by regularity theory, it remains zero for all times $t > T$. There are several explicit examples of solutions presenting the extinction phenomenon. One of them is the following separate variable solution

$$U(x,t;T) = k(m)\left(\frac{T-t}{|x|^2}\right)^{\frac{1}{1-m}}, \quad k(m) = 2\left(n - \frac{2}{1-m}\right)^{\frac{1}{1-m}}. \tag{1.16}$$

Moreover, the self-similar solutions called of *Type II*, having the form

$$u(x,t) = (T-t)^{\alpha} f(x(T-t)^{\beta})$$

present complete extinction in finite time if $\alpha > 0$.

The extinction phenomenon is analyzed and explained in great detail in the recent monograph by J. L. Vázquez, [127]. There, it is shown that indeed the self-similar solutions of Type II are essentially the typical solutions that represent the evolution process in this range (playing the role of the Barenblatt family for $m > m_c$ or $p > p_c$), thus EFT is a typical phenomenon of this range. There exist also precise extinction rates of the solutions, and properties of the extinction time T, as for example its continuous dependence on the initial data, are known.

Another phenomenon that characterizes the very fast diffusion range is the possible lack of regularity of solutions. Indeed, for $m < m_c$ in the PME and $p < p_c$ in the PLE, the $L^1 - L^\infty$ smoothing effect does not hold true; that is, solutions having an initial data $u_0 \in L^1$ may not become bounded immediately. In order to obtain immediate boundedness, we need to ask to the initial data u_0 to belong to a better functional space, L^r, $r > r_c$, for some critical exponent r_c (precise values are given in Chapter 4). Moreover, an even stronger failure of regularization holds, exemplified by the *backward effect*: if the initial data $u_0 \in L^r$ for $r < r_c$, then the solution will be only L^1 for positive times.

Finally, in recent papers the even more singular range $m \leq 0$ of the PME and $p < 1$ of the PLE are studied; they are usually called the *very fast diffusion range* or the *superfast diffusion range*. In this range, the mathematical phenomena are even more surprising, as for example the nonexistence of any solution with integrable (L^1) initial data.

In the present work, we will meet all these interesting phenomena in Chapter 2 and Chapter 4, that are in essential part dedicated to the Fast Diffusion. For a complete presentation of them, I recommend the recent book [127].

1.2 Structure and description of the thesis

The memoir contains four different parts, dealing with four problems related to the two basic Nonlinear Diffusion Equations, with special emphasis on the PLE. In the four main chapters, we deal with many typical problems related to this type of equations: special solutions, asymptotic behavior, regularity, Harnack inequalities and study of the interface behavior. We also study the influence of an extra term, in the form of a power of the gradient, in the most difficult case, that of the critical (resonant) exponent. But let us now describe in more detail the subject and the results obtained in this thesis.

(a) **Chapter 2** deals with the study of radially symmetric and, in particular, self-similar solutions of both equations. The importance of these special solutions for the qualitative theory is fundamental and discussed in the previous subsection.

The main result of Chapter 2 is clearly the very interesting and simple, explicit relation transforming the radially symmetric solutions of the PME into radially symmetric solutions of the PLE and conversely. Such a relation is very simple in dimension $n = 1$, since the relation in this case is given only by direct integration in the space variable. Moreover, there were previously in the literature many remarks about similar properties of the PME and PLE flows also in several dimension, that motivated our search for a general explicit formula explaining this.

Our relation generalizes directly the direct integration that has been already known in dimension $n = 1$. It is also essentially obtained by integration, but in several dimensions it also involves a change of dimension from the PME to the corresponding PLE, and multiplying by a power of the independent radial variable.

In the rest of the chapter, we apply these transformations and some techniques from Dynamical Systems (mainly *phase-plane analysis*) to study and classify the self-similar solutions of both equations, that is, solutions of the PLE or PME having one of the following three general forms:

$$\begin{cases} u(x,t) = t^{-\alpha} f(xt^{-\beta}), \\ u(x,t) = (T-t)^{\alpha} f(x(T-t)^{\beta}), \\ u(x,t) = e^{-\alpha t} f(xe^{-\beta t}), \end{cases} \tag{1.17}$$

where f is called *the profile* of the solution, and that are called self-similar solutions of type I, II and III respectively.

In particular, we obtain new interesting self-similar solutions of the PLE that were completely unknown before, such as the Dipole-type solution for the PLE (that will be used in Chapter 3 to study the asymptotic behavior in domains with holes) and the anomalous solutions for the Fast PLE. We are also able to find a *Hulshof-type sequence* of exponents giving rise to compactly supported self-similar solutions of the PLE, following from the celebrated series obtained by Hulshof in [75] for the PME. The last part of the chapter is devoted to a complete study of self-similar solutions for the Super-fast Diffusion range ($m < 0$ for the PME and $p < 1$ for the PLE), which is interesting but almost unexplored until now.

The results in Chapter 2 correspond essentially to those published in the papers [78] and [79].

(b) In **Chapter 3**, we study the asymptotic behavior of the solutions of the homogeneous Dirichlet problem for the PLE in a domain with holes. The study has as starting point the similar results obtained by C. Brändle, F. Quirós and J. L. Vázquez in [37] for the PME posed in a domain with holes. More precisely, we deal with the following problem:

$$\begin{cases} u_t = \Delta_p u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.18}$$

where $p > 2$ and $\Omega = \mathbb{R}^n \setminus G$ is the complement of a bounded open set. We give precise assumptions on the data at the beginning of the corresponding chapter.

Reducing the problem to the simplest one of an exterior domain, and following similar general ideas as in the PME case, the analysis of this problem has to be divided into three cases, with respect to the relation between the dimension n and the exponent p. We thus have three cases:

(i) For $n > p$, we actually prove that the holes do not play a very important role in the asymptotic behavior. This is the easiest case, and the important technical fact is that, if we shrink the hole to zero by a standard scaling process, in the end the pointwise singularity at the origin is removable. This allows for using the general techniques valid in the whole space, essentially the well-known Four-Step Method (see [128]). Thus, the asymptotic behavior is given in the outer region (situated far from the holes) by one representative of the Barenblatt family (1.11); the limit profile is unique when the initial datum u_0 is given, the uniqueness of it being proved by analyzing the mass of the limit profile.

(ii) For $n = p$, we are in the critical (resonant) case. As usual in such cases, some logarithmic corrections both in the decay rate and in the support expansion rate are expected to appear. In order to guess them, we adapt first a formal calculation coming from Gilding and Gonzerkiewicz [69] leading to the exact rates. We find that the asymptotic profile is in this case given not by a solution of the PLE or some stationary part of some scaled version of it, but by a subsolution of it, which appears as a Barenblatt solution, but with some logarithmic rates; more precisely, the limit profile is:

$$U(x,t;C) = (t\log\,t)^{-\frac{1}{p-1}}\left(C - k|x|^{\frac{p}{p-1}}(\log\,t^{\frac{p-2}{(p-1)^2}}\right)_+^{\frac{p-1}{p-2}}.$$

To prove in a rigorous manner the asymptotic convergence to some (unique when the initial datum is given) profile $U(x,t;C)$, we need stronger techniques, since the standard Four-Step Method fails here. In change, we can apply for this problem the stability technique developed by Galaktionov and Vazquez in [62]. In order to be able to use it, we need a first *continuous rescaling* step, taking into account the exact logarithmic scale introduced by the resonance and transforming the equation into a p-Laplacian nonlinear Fokker-Planck with an asymptotically small perturbation. We remark here (as in many other problems in PDE and in Applied Mathematics in general) how the understanding of the logarithmic correction and their correct guess are essential in order to apply correctly the rigorous technique and to prove the (already expected after the formal calculations) result.

(iii) The low dimensional case, $n < p$, is clearly the most interesting, difficult and unexpected of all three. Due to the small dimension, the hole has a dramatic influence, and the singularity it creates at the origin in the limit after the standard rescaling becomes essential. Thus, we are led to think about some self-similar profile of the PLE having a singularity at $x = 0$.

Having already in mind the complete study of self-similar solutions performed in Chapter 2, we find that such a candidate family of profiles exists precisely in dimensions $n < p$ and it is the so-called *dipole-type family* of the PLE. This is a completely new family of profiles of the PLE, discovered by our analysis in Chapter 1, that rescale in the following way:

$$D_\lambda(x,t) = t^{-\alpha}F(xt^{-\beta}), \quad F_\lambda(\eta) = \lambda^p F(\lambda^{2-p}\eta), \quad \forall\,\lambda > 0, \tag{1.19}$$

where the self-similarity exponents satisfy the relation:

$$(p-2)\alpha + p\beta = 1, \quad \alpha > 0, \quad \beta > 0, \tag{1.20}$$

but both exponents and the profile F_λ do not have explicit formulas. Moreover, these profiles are called *anomalous*, since they are not obtained by a conservation law, but as a special orbit in the phase-plane associated to some ODE. We will explain in more detail this in Chapter 3. Moreover, we obtain that $F_\lambda(0) = 0$, but its derivative is singular at $\eta = 0$. More precisely, near $\eta = 0$ we have

$$F(\eta) \sim \eta^{(p-N)/(p-1)}, \quad \text{as} \quad \eta \sim 0. \tag{1.21}$$

We prove that, indeed, this family of anomalous self-similar solutions of the PLE gives the outer asymptotic behavior in an exterior domain in dimension $n < p$. In this case we use a different technique in the proof, that of considering optimal barriers, with precedents in literature in papers like [61, 88]. In our case, its application becomes more complicated due to the non-existence of an explicit formula of the postulated limit profile and due to the failure of the Strong Maximum Principle. Indeed, the proof relies on a series of nontrivial geometrical and topological remarks, and a very careful and delicate *contact analysis*, whose idea is to eliminate the possible contacts between the limit profile and the optimal barrier from below, in case of non-coincidence of them. This contact analysis, based on the existence of suitable Harnack inequalities and/or intermediate barriers, is the complicated argument replacing the Strong Maximum Principle at the degenerate points of such a dipole-type solution.

The material of Chapter 3 is essentially published in the papers [80] and [81].

(c) In **Chapter 4**, we perform a more theoretical study of the PLE in its Fast Diffusion range, $1 < p < 2$, with emphasis on the regularity of solutions and on the expansion of positivity from the initial time to later times.

It is well-known, from the work of DiBenedetto, Gianazza, Vespri, Urbano etc. (see for example the survey [57]) that the standard PLE has very good regularizing properties in the range $p > 2$. More precisely, in this range any local weak solution with initial trace $u_0 \in L^1_{loc}(\Omega)$ for some domain $\Omega \in \mathbb{R}^n$, is immediately regularized, in the sense that $u \in C^{1,\alpha}$ at any time $t > 0$. In fact, this very complicated proof is performed in two steps: in a first step, a general regularity result shows that the solutions that are locally bounded are in fact Hölder continuous; then, in a second step, one can prove that a solution with initial trace $u_0 \in L^1_{loc}$ becomes automatically locally bounded. Moreover, estimates giving bounds of the local L^∞ norm in terms of local L^p norms, called *local smoothing effects*, are given, and optimal constants are calculated. Then, bounds from below, that represent the persistence and expansion of positivity when starting with an initial data supported in some ball, are given, and both estimates can be mixed to give various forms of *Harnack inequalities* (see [57]).

However, things become much more complicated when $p < 2$, and specially when p approaches 1. An essential critical point, as explained in the previous section, is $p_c = 2n/(n+1)$. In the Fast Diffusion Range $p < 2$ and specially for $p < p_c$, the known estimates in the form of local smoothing effect are not optimal, both qualitatively and quantitatively; moreover, an optimal estimate of the expansion of positivity, leading to a local bound from below for any local weak solution u, was missing in this range, and this concluded in the open problem proposed by DiBenedetto and his collaborators, to find a suitable form for Harnack inequalities for $p < p_c$ ([55, 54]). In Chapter 4, we concentrate on this open problem and we propose a solution to it.

A. *Optimal local smoothing effects.* In a first part, we are able to establish new and optimal local smoothing effects, showing, as already known in the PME case, that the regularization, for $p < p_c$, holds true if we start with an initial trace not in L^1_{loc}, but in a better space L^r_{loc}, where $r > r_c = n(2-p)/p$. We obtain also optimal local estimates of the L^∞ norm of u in terms of the local L^r norm, with exact optimal constants.

B. *Large solutions.* We show the existence of large solutions in a bounded domain, that is, solutions that are bounded in the interior of the domain and that tend to $+\infty$ when approaching the boundary. We are able to give the exact asymptotic range of approaching $+\infty$ close to the boundary. This result is new and, besides the interest by itself, it provides a very important class of solutions for further applications. In fact, during all Chapter 4, we will use the Large Solutions as comparison functions.

C. *Persistence of positivity.* These are lower estimates for the local solution u in terms of some local norms of its initial trace, having the general form

$$\inf_{x\in B_R(x_0)} u^{p-1}(x,t) \geq C R^{p-n} t^{\frac{p-1}{2-p}} \int_{B_R(x_0)} u_0(x)\, \mathrm{d}x,$$

for any $0 < t < t^*$. In particular, these inequalities play the role of a lower-Harnack, and they may be used to estimate, if we start with some quantity of fluid concentrated in a small ball, for how long and in which quantity the fluid stays in the initial ball. This justifies the given name. We obtain also a precise formula for the critical time t^* and improved formulas, with exact constants and optimal dependence, for the range $1 < p < p_c$ which is the most interesting in this line of research.

D. *Intrinsic Harnack inequalities.* Finally, just by joining the optimal upper bounds (smoothing effects) obtained in part A with the lower bounds in part C, we establish various forms of generalized Harnack inequalities. In these forms, as usual in the case of Fast Diffusion Equations, the constants are not global, they depend usually on quantities related to the same function; this is not allowed, thus, in order to get a more interesting and useful form, we have to accept that the parabolic cylinders where the inequality holds true depend on their center. This is the intrinsic geometry associated to the equation, and passing to this setting we can obtain Harnack inequalities in their standard form. We will be more precise within Chapter 4.

E. *The Special Energy Inequality.* This is an extremely short, natural, beautiful and interesting inequality that we find and whose applications we explore. If u is a continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q_T = \Omega \times (0,T)$, with $1 < p < 2$, and $0 \leq \varphi \in C_c^2(\Omega)$ is any admissible test function, then $u_t = \Delta_p u \in L^2_{loc}(Q_T)$ and the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x + \frac{p}{n}\int_\Omega (\Delta_p u)^2 \varphi \,\mathrm{d}x \leq \frac{p}{2}\int_\Omega |\nabla u|^{2(p-1)} \Delta\varphi \,\mathrm{d}x, \tag{1.22}$$

in the sense of distributions in $\mathcal{D}'(0,T)$. Various applications of it, in regularity or in estimates for the gradients of the solutions, are deduced and explored.

The results of this chapter were obtained in collaboration with Matteo Bonforte and are published in the paper [33].

(d) In **Chapter 5**, we study the asymptotic behavior and the interface expansion for the PLE with gradient absorption. More precisely, we investigate the asymptotic behavior for the following Cauchy problem

$$\begin{cases} \partial_t u - \Delta_p u + |\nabla u|^q = 0, & (x,t) \in Q, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.23}$$

with initial data $u_0 \geq 0$, compactly supported, and $p > 2$. The mathematical interest of the problem (besides of its interest in applications) is to study the influence of the gradient term on the evolution; generally, the presence of gradient terms can complicate the evolution and, balancing the effects of the diffusion and of the gradient term, one can obtain a very interesting and unexpected behavior.

In this case, the effect of the gradient term depends, of course, on the exponent q. For q very large, one expects that the diffusion term dominates in the evolution; on the contrary, for q small, the gradient term acts strongly and one expects it to dominate above the diffusion term. This also suggests the existence of critical, intermediate exponents q where unusual things could happen.

Indeed, works on regularity of solutions and estimates on the gradients for both the linear case $p = 2$ and the quasilinear case $p > 2$ (see [20, 64, 18]) identified two critical exponents:

$$q_1 = p - 1, \quad q_2 = p - \frac{n}{n+1}, \tag{1.24}$$

which delimitate different aspects in the asymptotic behavior. In the case $1 < q < p - 1$, the problem has been solved by Laurençot and Vázquez in [99], where they proved that the diffusion term is negligible in the asymptotic limit and the gradient absorption dominates. The asymptotic profile is thus a fixed *sandpile*, obtained as a solution of the Eikonal-type Hamilton-Jacobi equation that remains after just eliminating the diffusion.

The original result of this Chapter regards the same problem of asymptotic behavior and interface expansion for the critical case $q = p - 1$, where similar techniques as in [99] do not work. This case determined a delicate and technically complicated analysis, trying to balance, as usual in resonant cases, the effect of the two processes competing in the equation.

In this case, the limit profile has a complicated form, being in simplified scaling variables again a kind of sandpile (non-regular at the origin), with a cusp on top. It is obtained as the solution of a stationary Hamilton-Jacobi equation, but after several scaling steps. The technique of the proof is complicated and relies in two different scaling steps and some technically difficult construction of suitable subsolutions starting from a traveling wave analysis. Thus, in the proof many elements of deep mathematics combine. Moreover, the comparison with traveling wave subsolutions and a first scaling step (passing to *logarithmic time*) allows to establish the interface behavior, which, as expected, has a logarithmic rate of advance.

The results in this chapter are taken from the paper [77].

Chapter 2

Radial equivalence and special solutions for the two basic nonlinear diffusion equations

In this first chapter, we present some new and very interesting transformations between the radially symmetric solutions of the PME and of the PLE. We apply our transformations in order to discover new special solutions of both equations that were unknown before.

2.1 Main results for $m > 0$ and $p > 1$

It has been observed since many years that the theory of the two equations we consider has many parallel results, concerning the finite speed of propagation in the case of slow diffusion, the asymptotic behavior of the general solutions, and the existence of some special solutions, called self-similar solutions, which play an important role in the theory, cf. for instance [52], [129], and [132].

Maybe the clearest connection between both equations takes place in space dimension $n = 1$ where the PLE is obtained through a formal integration of the PLE, see for example [24]:

$$\bar{u}(x,t) = \int_{-\infty}^{x} u(s,t)ds + c, \tag{2.1}$$

where $\bar{u}$ represents a solution of the PLE, u is a solution of the PME and c is an arbitrary constant[1]. The plan of this chapter is to extend the equivalence of both equations to arbitrary dimensions under the condition of radial symmetry. We will work first in the usual ranges $m > 0$ and $p > 1$. Extensions to $m \leq 0$ and $p \leq 1$ are interesting and have appeared in the recent literature, and we devote the second part of the present chapter to this range.

[1] We do not need to start integrating from $x = -\infty$, but then the integrand is more complicated, cf. [114], pp. 144–145.

We make two remarks before presenting the statements: first, in all this chapter we consider that the range of the dimensions is continuous. Although from the physical point of view this does not make sense if the dimension is not an integer, in the setting of radial solutions this can be allowed, since the dimension appears only as a parameter in the radial formulation of the equations; we will only restrict it to be positive. Second, we will use in all the text the notation with bar for variables or parameters in the PLE case. Thus, we will denote by $r=|x|$ in the PME case and $\bar{r}=|x|$ in the PLE case.

The analysis will be different in dimensions $0<n<2$ and in dimensions $n>2$ in the PME case. The main correspondence relations between radially symmetric solutions in the ranges $m>0$ and $p>1$ are:

Theorem 2.1. *Suppose $0<n<2$. Then the radially symmetric solutions u and $\bar{u}$ of the PME, resp. PLE, are related through the following transformation:*

$$\bar{u}_{\bar{r}}(\bar{r},t)=D_1 r^{\frac{2n-2}{m+1}}u(r,t), \quad D_1=\Big(\frac{(mn-n+2)^2}{m(m+1)^2}\Big)^{\frac{1}{m-1}}, \tag{2.2}$$

where the correspondence of the parameters is

$$p=m+1, \quad \bar{n}=\frac{(n-2)(m+1)}{n-mn-2} \tag{2.3}$$

and the independent variables are related by $\bar{r}=r^{(mn-n+2)/(m+1)}$.

Note that now $\bar{n}$ is a monotone decreasing function of n for fixed m, and ranges from $p=m+1$ to 0 while n goes from 0 to 2.

Theorem 2.2. *Suppose $2<n<\infty$. Then the radially symmetric solutions u and $\bar{u}$ of the PME, resp. PLE, are related through the following transformation:*

$$\bar{u}_{\bar{r}}(\bar{r},t)=D_2 r^{\frac{2}{m+1}}u(r,t), \quad D_2=\Big(\frac{(2m)^2}{m(m+1)^2}\Big)^{\frac{1}{m-1}} \tag{2.4}$$

where the correspondence of the parameters is

$$p=m+1, \quad \bar{n}=\frac{(n-2)(m+1)}{2m} \tag{2.5}$$

and the independent variables are related by $\bar{r}=r^{2m/(m+1)}$.

In this case $\bar{n}$ is a linear increasing function of n for fixed m, and ranges from 0 to infinity while n goes from 2 to infinity.

Further remarks: (i) For $n=1$ we recover the already known equivalence transformation (2.1). Note that the radiality condition is not needed.

(ii) For $n=2$, the previous formulas formally give $\bar{n}=0$ and the correspondence relations are identical in both cases. If the dimension $\bar{n}=0$ is accepted, the proofs of the main results

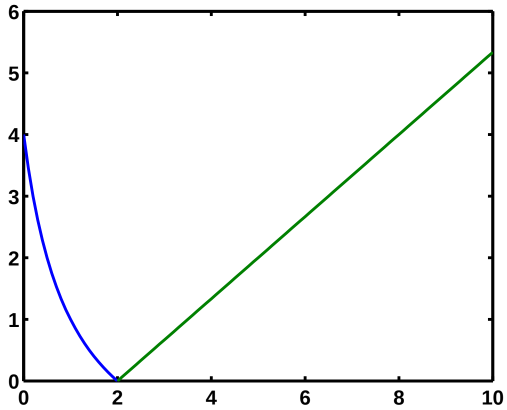

Figure 2.1: The dependence of $\bar{n}$ on n, for $m = 3$.

and all the calculations are similar, but we will not go into this case because we think it brings nothing new.

(iii) We stress the fact that in general the correspondence implies change of the spatial dimension. Indeed, dimension is conserved in the first branch only if $n = 1$, and in the second if $m = m_s := (n-2)/(n+2)$. Moreover, for every $m > 0$ there are two options for the equivalence maps from PME into the same PLE, and that will give rise to a self-map of the PME, that we describe in Section 2.3. These branches are represented in the Figure 2.1.

Self-similar solutions

We will examine in detail the application of the equivalence to the class of special solutions of self-similar type. This is important in the theory since it is well-known that self-similar solutions play a fundamental role in discovering the main properties to be expected from the theory of nonlinear equations with good scaling properties, and, once identified, they are used in describing the main qualitative and asymptotic results for wide classes of general solutions. On the other hand, the motivation for the discovery of the above transformations came from the study of self-similar solutions. More precisely, the phase-plane analysis of the self-similar solutions that we will present in Section 2.4 was the way to identify the curious values of the exponents in the transformations of Theorems 2.1 and 2.2.

For the PME several fundamental families of self-similar solutions are known. Maybe the most important one is formed by the Barenblatt solutions, discovered independently by Barenblatt in [11] and by Zeldovich and Kompaneets in [134], which are:

$$B_C(x,t) = t^{-\alpha}\left(C - k(\frac{|x|}{t^{\beta}})^2\right)_+^{\frac{1}{m-1}}, \tag{2.6}$$

where $C > 0$ is a free parameter, and α, β and k have precise values:

$$\alpha = \frac{n}{n(m-1)+2}, \quad \beta = \frac{1}{n(m-1)+2}, \quad k = \frac{m-1}{2(n(m-1)+2)}. \tag{2.7}$$

Note that this definition applies for $m > m_c := (n-2)/n$. There is a similar family for $m < m_c$ but the form and the properties are quite different, cf. [127] and [30].

For the PLE there is an equivalent family of self-similar solutions, called also Barenblatt for being similar to the first ones, and having the explicit form:

$$U_C(x,t) = t^{-\alpha}\left(C - k(\frac{|x|}{t^{\beta}})^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{2.8}$$

where the constants are

$$\alpha = \frac{n}{n(p-2)+p}, \quad \beta = \frac{1}{n(p-2)+p}, \quad k = \frac{p-2}{p}\beta^{\frac{1}{p-1}}. \tag{2.9}$$

A second family consists of the dipole-type solutions, discovered by Barenblatt and Zeldovich in [15] and generalized by J. R. King in [93], and having the explicit form:

$$Z_C(x,t) = t^{-\alpha}U(xt^{-\beta}), \quad U(\eta) = \pm|\eta|^{\frac{2-n}{m}}\left(C - \frac{m-1}{2(n(m-1)+2)}|\eta|^{n+\frac{2-n}{m}}\right)_+^{\frac{1}{m-1}}, \tag{2.10}$$

where the exponents are $\alpha = \frac{1}{m}$, $\beta = \frac{1}{2m}$ independently on the dimension. These solutions coincide with the Barenblatt solutions for $n = 2$ and become singular at $x = 0$ for $n > 2$. We will discuss below what is the equivalence of this family for the PLE. We will also call in the sequel the constant C appearing in the definitions of these particular solutions as their free parameter.

All these solutions are particular cases of the general class of self-similar solutions, which will be our main application topic in this chapter. Self-similar solutions are broadly defined as those solutions $u(x,t)$ whose space profile $u(\cdot,t)$ is independent of time but for a possible time-dependent scaling (or zoom) in the variables u and x. It is well-known (see [131, Chapter 16] for a proof in the PME case) that for our equations the self-similar solutions can take one of the three following forms:

$$\begin{cases} u(x,t) = t^{-\alpha}f(xt^{-\beta}), \\ u(x,t) = (T-t)^{\alpha}f(x(T-t)^{\beta}), \\ u(x,t) = e^{-\alpha t}f(xe^{-\beta t}), \end{cases} \tag{2.11}$$

that we will call self-similar solutions of type I, II and III respectively. It is often said that type I describes forward self-similarity. The importance of these types of solutions lies in the fact that they usually describe the large-time behavior of general compactly supported solutions. Type II receives also the name of backward self-similarity; they are called in geometry *ancient solutions* since they exist since $t = -\infty$ but not necessarily for all positive times. Precisely, they are often used to describe phenomena of extinction (cf. [127]) or blow-up (in reaction diffusion equations, cf. [118]). The solutions of type III, also called exponential self-similarity, are important in the critical fast-diffusion case (cases $m = m_c := (n-2)/n$ and $p = p_c := 2n/(n+1)$ below).

After discovering the particular solutions just presented, an important work in the direction of discovering and classifying all the self-similar solutions of the two equations started, see for example [71], [72], [67], [75], [76], [7], [8], and [131] for the PME and fast-diffusion and, more recently, the works [27] and [28] for the PLE.

The results apply to the three types of self-similarity, and we will use in the sequel a parameter ε to select the type of solutions: indeed, $\varepsilon = 1$ for solutions of Type I, $\varepsilon = -1$ for solutions of Type II and $\varepsilon = 0$ for solutions of Type III. As we have already said, the next theorem was the first step to arrive to the general correspondence of the radial solutions.

Theorem 2.3. *(i) The analysis of the radial self-similar solutions for the PME (when $m \neq 1$) and for the PLE (when $p \neq 2$) can be reduced to the analysis of a particular case of the autonomous ODE system*

$$\begin{cases} \dot{\Psi} = \Psi\Phi, \\ \dot{\Phi} = c_1\Phi^2 - c_2\Psi\Phi - c_3\Phi + \varepsilon\Psi + sgn(b), \end{cases} \tag{2.12}$$

where the coefficients c_1, c_2, c_3 and b are explicit functions of m, n, β in the PME case, respectively of p, $\bar{n}$ and $\bar{\beta}$ in the PLE case. The variables Φ and Ψ are given by explicit expressions in terms of η, f, $f^{'}$, respectively $\bar{\eta}$, $\bar{f}$, $\bar{f}^{'}$.

(ii) If $n \neq 2$, the correspondence is exact if the following equalities between the parameters hold:

$$p = m+1, \quad \bar{n} = \frac{(n-2)(m+1)}{n-2-mn}, \quad \bar{\beta} = \frac{mn-n+2}{m+1}\beta, \quad \bar{\alpha} = \frac{(mn-n+2)\alpha - n\varepsilon}{2}, \tag{2.13}$$

if $0 < n < 2$, or

$$p = m+1, \quad \bar{n} = \frac{(n-2)(m+1)}{2m}, \quad \bar{\beta} = \frac{2m}{m+1}\beta, \quad \bar{\alpha} = m\alpha - \varepsilon, \tag{2.14}$$

if $n > 2$.

The variables η and $\bar{\eta}$ in part (i) stand for $|x|\,t^{-\beta}$ in type I, for $|x|\,(T-t)^{\beta}$ in type II and for $|x|\,e^{\beta t}$ in type III, while over-dot in (2.13) indicates derivative with respect to a re-parametrization of η that may depend on the orbit.

The rest of the chapter will be devoted to applications and extensions of our results. We first deduce, in the particular case of the self-similar solutions, an improvement of the correspondence relations, by obtaining an explicit expression of the PLE profile $\bar{f}$ as a function of the PME profile f and its derivative $f^{'}$ (see Section 2.5). Next, we translate all that it is known in the case of one of the equations into the other via simple and direct calculations. In this way, in Section 2.6, as an important application, we obtain a Hulshof-type sequence of solutions (see [75], [76] and the definition at the beginning of Section 2.6) in the PLE case, in particular obtaining new solutions of dipole-type for the PLE, which, differently from the well-known PME case (2.10), are not explicit. In the PME case, these solutions exist in the case $0 < n < 2$ and have physical sense only for $n = 1$, but in the PLE case they exist in $0 < \bar{n} < p$ and have physical sense for many dimensions. On the other hand, in Section 2.8, dedicated to the fast-diffusion case, after obtaining a similar Hulshof-type sequence in the supercritical case $p > p_c$, we derive a new branch of solutions of the PLE with anomalous exponents and having optimal decay at infinity in the subcritical case $p < p_c$, corresponding to the branch of solutions of the PME from [108] and [127].

Finally, there exists a second part of the present chapter devoted to the so-called very fast range, namely $m < 0$ for the PME and $p < 1$ for the PLE. In this range our correspondence relations still hold, in a slightly different form and manner, and we study them in detail starting from Section 2.9.

2.2 Proof of equivalence for radial solutions

Suppose that $u(r,t)$ is a radially symmetric solution of the PME. It satisfies

$$u_t = r^{1-n}\frac{\partial}{\partial r}(r^{n-1}|u|^{m-1}u_r). \tag{2.15}$$

Similarly, a radially symmetric solution $\bar{u}(\bar{r},t)$ of the PLE satisfies

$$\bar{u}_t = \bar{r}^{1-\bar{n}}\frac{\partial}{\partial \bar{r}}(\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{p-2}\bar{u}_{\bar{r}}). \tag{2.16}$$

(i) Suppose that $0 < n < 2$, u is a radially symmetric solution of the PME and $\bar{u}$ given by (2.2). Using the transformations in (2.3), we obtain

$$\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{p-2}\bar{u}_{\bar{r}} = D_1^m|u|^{m-1}u,$$

hence

$$\bar{r}^{1-\bar{n}}\frac{\partial}{\partial \bar{r}}(\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{p-2}\bar{u}_{\bar{r}}) = \frac{m+1}{mn-n+2}D_1^m r^{n-1}(|u|^{m-1}u)_r. \tag{2.17}$$

Note that since $0 < n < 2$ and $m > 0$, we have $mn - n + 2 > 0$. On the other hand, by differentiating with respect to time in (2.2), we obtain that

$$\bar{u}_{\bar{r},t} = D_1 r^{\frac{2n-2}{m+1}}u_t.$$

Hence, in order to finish the proof of Theorem 2.1, we have to differentiate again with respect to $\bar{r}$ in (2.17). After a straightforward calculation, we obtain

$$\frac{\partial}{\partial \bar{r}}\left(\bar{r}^{1-\bar{n}}\frac{\partial}{\partial \bar{r}}(\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{p-2}\bar{u}_{\bar{r}})\right) = D_1 r^{-\frac{(n-1)(m-1)}{m+1}}\frac{\partial}{\partial r}(r^{n-1}|u|^{m-1}u_r) = D_1 r^{\frac{2(n-1)}{m+1}}u_t.$$

We deduce that $\bar{u}$ is a radially symmetric solution of the PLE. The converse correspondence is similar.

(ii) In the case $2 < n < \infty$, we perform analogous calculations. To begin with, we have

$$\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{p-2}\bar{u}_{\bar{r}} = D_2^m r^{n-2}|u|^{m-1}u,$$

hence

$$\bar{r}^{1-\bar{n}}\frac{\partial}{\partial \bar{r}}(\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{p-2}\bar{u}_{\bar{r}}) = \frac{m+1}{2m}D_2^m r^{3-n}\frac{\partial}{\partial r}(r^{n-2}|u|^{m-1}u). \tag{2.18}$$

As in the first case, by differentiating with respect to time in (2.4), we find

$$\bar{u}_{\bar{r},t} = D_2 r^{\frac{2}{m+1}} u_t.$$

Hence, we have to differentiate again in (2.18) with respect to $\bar{r}$. After performing straightforward calculations, we arrive to

$$\frac{\partial}{\partial \bar{r}}\Big(\bar{r}^{1-\bar{n}}\frac{\partial}{\partial \bar{r}}(\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{p-2}\bar{u}_{\bar{r}})\Big) = D_2 r^{\frac{(1-n)(m+1)+2}{m+1}}\frac{\partial}{\partial r}(r^{n-1}|u|^{m-1}u_r) = D_2 r^{\frac{2}{m+1}} u_t.$$

This shows that $\bar{u}$ is a solution of the PLE. The converse correspondence is similar.

We remark a common feature of both cases: since at the final step we have to integrate, from a single solution of the PME we obtain through these correspondence relations not a single solution of the PLE, but any solution of the form $\bar{u} + C$, with $C \in \mathbb{R}$. Of course, invariance under addition of a constant is a well-known property of the solution set of the PLE.

The previous computations are true at the formal level, in particular when the solutions of the PME are C^2, which happens whenever $u \neq 0$. The conclusion extends to the more general class of weak solutions (with C^α regularity for the PME and $C^{1,\alpha}$ for the PLE). It is also valid for more general solutions with some types of singularities. We will refrain at this point from entering into the cumbersome question of justification. Instead, the analysis of the classes of self-similar solutions will allow us to consider the most important types of non-classical solutions.

2.3 Inverting the correspondence. Self-maps

If we want to go back from the PLE to the PME, we see from Figure 2.1 that for any dimension $\bar{n} > 0$ and $p \neq p_c$, there are for most $m's$ two possible values of n, denoted by n_1 and n_2 and related by the formula:

$$\frac{1}{n_1 - 2} + \frac{1}{n_2 - 2} = \frac{1-m}{2m}. \tag{2.19}$$

Note that both branches coincide when $n_1 = n_2 = 2$; on the other hand, $n_2 \geq 2(1+m)$ corresponds to $n_1 \leq 0$, which we do not consider. We conclude that given $m > 0$ there is a self-map of the PME given by change of dimension defined for $n_1 \in (0,2)$ with values $2 < n_2 < 2(m+1)$. The formula is

$$n_2 = 2 + \frac{(2-n_1)2m}{2 + mn_1 - n_1}$$

Note that in both ranges $m > m_c$, i.e., $n_i(m-1)+2 > 0$. Note also that the interval of values of $\bar{n}$ that allows for two branches is $(0,p)$. The self-map for the PME has been remarked also by J. King in [94] and [95]. We derive the explicit correspondence between the radially symmetric solutions in dimensions n_1 and n_2. By equating the correspondence relations of the independent variables r_1 and r_2 with the same $\bar{r}$, we obtain that

$$r_2 = r_1^{\frac{mn_1 - n_1 + 2}{2m}},$$

On the other hand, by equating the correspondences between u and $\bar{u}$ in dimensions n_1 and n_2, we have

$$D_2 r_2^{\frac{2}{m+1}} u_2(r_2,t) = D_1 r_1^{\frac{2n_1-2}{m+1}} u_1(r_1,t),$$

hence

$$u_2(r_2,t) = \frac{D_1}{D_2} r_1^{\frac{n_1-2}{m}} u_1(r_1,t), \tag{2.20}$$

where we denote by u_i the solution in dimension n_i, $i = 1, 2$. We will exemplify the way this transformation acts in the case of self-similar solutions, in the proof of Theorem 2.4 in Section 2.6.

In order to look for self-maps of the PLE case, we have to examine the relation that joins the two possible values of $\bar{n}$ for each value of n. It is

$$\frac{1}{\bar{n}_1} + \frac{1}{\bar{n}_2} = \frac{2-p}{p} = \frac{1-m}{m+1}. \tag{2.21}$$

We deduce that $1/\bar{n}_1 < (2-p)/p$, i.e. the only case where the self-map could appear is for $p < p_c = 2\bar{n}/(\bar{n}+1)$. But in this case, the correspondence of the dimensions is given only by

$$\bar{n} = \frac{(n-2)(m+1)}{2m},$$

which is obviously a bijection. Hence there are no interesting self-maps for the PLE.

Remarks: (i) In the limit case $m = 1$, we obtain a self-map for the heat equation, which makes physical sense in the case $n_1 = 1$, $n_2 = 3$. In this case $r_1 = r_2$ and the correspondence between the radially symmetric solutions becomes

$$u_2(r,t) = \frac{1}{r} u_1(r,t). \tag{2.22}$$

This relation has also appeared in King's [95]. The reader is asked to make a direct verification of this easy fact. More generally, for $m = 1$, we obtain a correspondence between the heat equation in dimensions $\bar{n}$ and n, where $\bar{n} = \pm(n-2)$, where the signs are such that $\bar{n} > 0$. Here $\bar{r} = r$ and the relation between the radial solutions becomes

$$\bar{u}_r(r,t) = r u(r,t). \tag{2.23}$$

In Section 2.7 we will show how this correspondence acts on explicit examples of solutions;

(ii) By fixing $n_1 = 1$ and an integer number $n_2 > 2$ and trying to find an appropriate m in order to correspond n_1 to n_2, we find that $m = (n_2-2)/(4-n_2)$, which is nonpositive for $n_2 \geq 4$. Hence, the unique self-map between two integer dimensions is that of $n_1 = 1$, $n_2 = 3$ and $m = 1$, i.e. for the heat equation, as described in the first remark.

2.4 Phase-plane analysis of self-similar solutions

In this section we introduce the phase-planes associated to the self-similar solutions of the PME and of the PLE and we prove Theorem 2.3.

2.4.1. Phase-plane for the PME. In the PME case, the phase-plane analysis is well-known and described in detail in [131], Chapter 16, where references are given. First, the relation between the similarity exponents α and β reads $(m-1)\alpha+2\beta=\varepsilon$ where $\varepsilon=+1,-1$, or 0 depending on the type. Then, under the usual assumption of radial symmetry, the profile f must satisfy the ODE

$$\eta^{1-n}(\eta^{n-1}|f|^{m-1}f')'+\alpha f+\beta\eta f'=0, \tag{2.24}$$

where $\eta>0$. We now introduce the variables:

$$X=\eta f'/f, \qquad Y=\eta^2|f|^{1-m}. \tag{2.25}$$

Note that Y is nonnegative even if the solution changes sign. After replacing the η variable by $r=\log\eta$, the functions $X(r)$ and $Y(r)$ satisfy the classic autonomous ODE system (see for example [131]):

$$\left\{\begin{array}{l}\dot{X}=(2-n)X-mX^2-(\alpha+\beta X)Y,\\ \dot{Y}=(2+(1-m)X)Y,\end{array}\right. \tag{2.26}$$

where over-dot indicates differentiation with respect to r. We need to perform a further transformation of the variables, in order to obtain an easier phase-plane. Assuming furthermore that $m\neq m_c:=(n-2)/n$, we introduce a new pair of variables

$$\Phi=(2+(1-m)X)/\sqrt{|b|}, \qquad \Psi=Y/|b|, \tag{2.27}$$

where $b=2n(m-m_c)/(m-1)\neq 0$. In these variables, the system becomes

$$\left\{\begin{array}{l}\dot{\Psi}=\Psi\Phi,\\ \dot{\Phi}=c_1\Phi^2-c_2\Psi\Phi-c_3\Phi+\varepsilon\Psi+\operatorname{sgn}(b),\end{array}\right. \tag{2.28}$$

where we have replaced the r variable by $r_1=\sqrt{|b|}r$, so that over-dot indicates now differentiation with respect to r_1. Therefore, the system takes the desired quadratic form (2.28), with precise values for the constants given by

$$c_1=\frac{m}{m-1}, \quad c_2=\beta\sqrt{|b|}, \quad c_3=\frac{(n+2)(m-m_s)}{(m-1)\sqrt{|b|}}, \quad m_s=\frac{n-2}{n+2}. \tag{2.29}$$

With these values System (2.28) has free parameters m, n and β, since α can be calculated from them. The case $m=m_c$ will be discussed below.

2.4.2. Phase-plane for the PLE. The transformations in the PLE case are more involved and we will describe them in detail. For simplicity, we will skip in this subsection the notation with bar that we have adopted as a rule for the PLE case. First of all, the relation between the similarity exponents α and β becomes $(p-2)\alpha+p\beta=\varepsilon$. The radially symmetric profile f satisfies the ODE

$$\eta^{1-n}(\eta^{n-1}|f'|^{p-2}f')'+\alpha f+\beta\eta f'=0, \tag{2.30}$$

where prime denotes differentiation with respect to $\eta>0$. In a similar way as before, for $p\neq 2$ we introduce phase plane variables, a bit different from the ones in the PME case:

$$X=-\eta^2|f'|^{1-p}f', \qquad Z=\eta^\gamma f, \quad \text{where} \quad \gamma=\frac{p}{2-p}. \tag{2.31}$$

From (2.31) we obtain

$$|f'| = (\eta^{-2}|X|)^{\frac{1}{2-p}}, \quad f' = -\eta^{\frac{2}{p-2}}|X|^{-\frac{p-1}{p-2}}X. \tag{2.32}$$

This implies that

$$\begin{aligned}(\eta^{n-1}|f'|^{p-2}f')' &= (-\eta^{n+\frac{p}{p-2}}|X|^{-\frac{2p-3}{p-2}}X)' \\ &= \frac{p-1}{p-2}\eta^{n+\frac{p}{p-2}}|X|^{-\frac{2p-3}{p-2}}X' - (n+\frac{p}{p-2})\eta^{n+\frac{2}{p-2}}|X|^{-\frac{2p-3}{p-2}}X.\end{aligned} \tag{2.33}$$

If we substitute (2.31), (2.32) and (2.33) into (2.30), we get the autonomous ODE system

$$\begin{cases} \frac{p-1}{2-p}\dot{X} = -(n-\gamma)X + \alpha Z|X|^{\frac{3-2p}{2-p}} - \beta|X|X, \\ \dot{Z} = \gamma Z - |X|^{\frac{p-1}{2-p}}X, \end{cases} \tag{2.34}$$

where we have replaced the η variable by $r = \log\eta$, so that over-dot indicates differentiation with respect to r. This system is not quadratic, so that we perform a further change with this objective in mind. We introduce the new variable $Y = |X|^{\frac{1}{p-2}}XZ = -\eta|f'|^{-p}f'f$ and the flow equations become

$$\begin{cases} \dot{X} = \frac{2-p}{p-1}X\,(\gamma - n + \alpha Y - \beta|X|) \\ \dot{Y} = \frac{p-1}{p-2}|X|^{\frac{1}{p-2}}X'Z + |X|^{\frac{1}{p-2}}XZ' = -\alpha Y^2 + nY + \beta Y|X| - |X|, \end{cases} \tag{2.35}$$

which is a quadratic system if X has a sign. For the next step, we set:

$$\Psi = a|X|, \qquad \Phi = -\frac{p-2}{(p-1)\sqrt{|b|}}\,(\gamma - n + \alpha Y - \beta|X|)\,. \tag{2.36}$$

If we substitute (2.36) in (2.35) we obtain in the first equation

$$\dot{\Psi} = \sqrt{|b|}\Psi\Phi$$

and in the second equation

$$\dot{\Phi} = \sqrt{|b|}(c_1\Phi^2 - c_2\Psi\Phi - c_3\Phi + c_4\Psi + c_5),$$

where

$$\begin{aligned} c_1 &= \frac{p-1}{p-2}, \quad c_2 = \frac{\beta}{(p-1)a\sqrt{b}}, \quad c_3 = \frac{n-2\gamma}{\sqrt{|b|}}, \\ c_4 &= \frac{(p-2)(\alpha-\beta\gamma)}{(p-1)a|b|}, \quad c_5 = \frac{(p-2)(\gamma-n)\gamma}{(p-1)|b|}. \end{aligned}$$

Now, if we equalize c_4 to ε and c_5 to ±1 we obtain

$$a = \frac{1}{|b|(p-1)}, \quad b = \frac{p(n+1)(p-p_c)}{(p-2)(p-1)}, \quad \text{where} \quad p_c = \frac{2n}{n+1},$$

where we have used that $\gamma = -p/(p-2)$ and the relation between the exponents, $(p-2)\alpha + p\beta = \varepsilon$. We remark that in the case of solutions of Type III, $c_4 = 0$ and we can choose any number a. For convenience, we will use the same value of a in this case. In order to continue, we need to assume that $p \neq p_c$. After all these transformations, the flow equations become exactly the desired (2.28) and the constants are now given by

$$c_1 = \frac{p-1}{p-2}, \quad c_2 = \beta\sqrt{|b|}, \quad c_3 = \frac{(n+2)(p-p_s)}{(p-2)\sqrt{|b|}}, \quad p_s = \frac{2n}{n+2}. \tag{2.37}$$

This system has free parameters p, n and β, and $p = 2$ is excluded. We have replaced the r variable by $r_1 = \sqrt{|b|}r$, so that over-dot indicates differentiation with respect to r_1.

2.4.3. The critical cases, m_c and p_c. Some changes have to be made in these special cases $m = m_c$ for the PME or $p = p_c$ for the PLE, since our definitions imply that $b = 0$. In these cases $n > 2$ and $\bar{n} > 0$ and b changes into $\sqrt{b} = n - 2$ for PME and $\sqrt{b} = n$ for PLE, so that $c_3 = -1$ and the independent term $\text{sgn}(b)$ disappears from the second equation of system (2.28), which becomes

$$\begin{cases} \dot{\Psi} = \Psi\Phi, \\ \dot{\Phi} = c_1\Phi^2 - c_2\Psi\Phi + \Phi + \varepsilon\Psi, \end{cases} \tag{2.38}$$

with c_1 and c_2 as before.

Summing up, our systems can differentiate between the three types of self-similarity through only the value of the coefficient of Ψ.

2.4.4. Correspondence of the parameters. We want to find the correspondence between the parameters (m, n, β) of the PME and the parameters $(p, \bar{n}, \bar{\beta})$ for the PLE so that the expressions of the coefficients of (2.28) and (2.37) are the same. From now on we will use the notation with bar for the PLE case in all the rest of the text.

- By identifying c_1, we obtain the usual and expected relation $p = m + 1$.
- Then, we identify the coefficients c_3 and we obtain

$$\frac{(n+2)(m-m_s)}{(m-1)|b|^{1/2}} = \frac{(\bar{n}+2)(p-p_s)}{(p-2)|\bar{b}|^{1/2}}. \tag{2.39}$$

Replacing b and $\bar{b}$ by the explicit expressions given above and putting $p = m+1$, we deduce that

$$2m(mn-n+2)(\bar{n}(m-1)+2(m+1))^2 = (m+1)(\bar{n}(m-1)+(m+1))(n(m-1)+2(m+1))^2$$

which is a quadratic expression on $\bar{n}$. Solving this equation, we obtain two possible branches of $\bar{n}$ as a function of n:

$$\bar{n}_1 = \frac{(m+1)(n-2)}{2m}, \qquad \bar{n}_2 = \frac{(m+1)(n-2)}{n-2-mn} = \frac{(m+1)(n-2)}{n(m_c-m)}, \tag{2.40}$$

represented in Figure 2.1. The two branches are different unless $m = 1$ (which is an exceptional case) or $n = 2$ (which needs a separate consideration since $\bar{n}_1 = \bar{n}_2 = 0$ unless

$m = m_c = 0$). More precisely, from (2.40) we deduce that the first value of $\bar{n}$ is positive only for $n > 2$. The second value is positive only for $n < 2$ when $m > m_c$, and is also positive for $n > 2$ if $m < m_c$. Apparently, in the case $m < m_c$ (where only $n > 2$ makes sense) we have two valid different branches of our correspondence. But if we replace in the explicit expression of $\bar{c}_3$ the value of $\bar{n}_2$, we obtain

$$c_3 = \frac{\sqrt{2}}{2}\frac{(mn+2m-n+2)\,\mathrm{sgn}(m-1)}{\sqrt{|(m-1)(mn-n+2)|}}, \quad \bar{c}_3 = \frac{\sqrt{2}}{2}\frac{(mn+2m-n+2)\,\mathrm{sgn}(m-1)}{(mn-n+2)\sqrt{|(m-1)/(mn-n+2)|}}.$$

We remark that $c_3 = -\bar{c}_3$ in this case. This shows that in the case $m < m_c$, $\bar{n}_2$ is not a solution of (2.39). The appearance of this false solution is explained by the fact that taking squares in (2.39), we include the possibility that $c_3 = -\bar{c}_3$, as it happens here. Hence in all the cases, the correspondence $\bar{n}_1$ holds for $n > 2$ and the correspondence $\bar{n}_2$ holds for $n < 2$ (in which case $m_c < 0$ so that $m > m_c$). That is why we will separate our analysis in two different cases, one with $n > 2$ and another one with $0 < n < 2$.

• We have to identify the coefficients c_2. This implies that $\beta|b|^{1/2} = \bar{\beta}|\bar{b}|^{1/2}$, hence

$$\frac{\beta^2}{\bar{\beta}^2} = \frac{(m-1)p(\bar{n}+1)(p-p_c)}{2n(m-m_c)(p-2)(p-1)}.$$

From here we deduce the correspondence of the exponents:

$$\bar{\beta}_1 = \frac{2m}{m+1}\beta = \frac{n-2}{\bar{n}_1}\beta, \quad \bar{\beta}_2 = \frac{mn-n+2}{m+1}\beta = \frac{2-n}{\bar{n}_2}. \tag{2.41}$$

We note that the first formula holds only for $n > 2$ and the second formula holds only for $n < 2$. We remark that in both cases $(n-2)^2\beta^2 = \bar{n}^2\bar{\beta}^2$. The same formulas (2.41), considering only the first equalities, hold also for $n = 2$ and coincide.

• We still have to check that $\mathrm{sgn}\, b = \mathrm{sgn}\, \bar{b}$. But this is true, since a straightforward calculation gives us that

$$\frac{b}{\bar{b}} = \begin{cases} \dfrac{(mn-n+2)^2}{(m+1)^2}, & \text{if } n < 2, \\[2ex] \dfrac{4m^2}{(m+1)^2}, & \text{if } n > 2. \end{cases}$$

This completes the proof of Theorem 2.3. We also remark that for $n = 1$ we obtain a simpler correspondence, since $\bar{n} = 1$ and $\bar{\beta} = \beta$. This corresponds to the well-known fact that if we differentiate the PLE in dimension 1 we obtain the PME.

This ends the analysis of the map from the PME to the PLE.

Remarks: (i) In the analysis above we have worked only with $m > 0$. If we accept negative values of m, i.e. we consider the case of very-fast diffusion, the things change a bit since the range is expanded. We deal with this range in the second part of this chapter, starting from Section 2.9.

(ii) We can identify the critical cases $m = m_c$ in the PME and $p = p_c$ in the PLE. In that case, $\bar{n} = n - 1$ and $\bar{\beta} = \beta(n-2)/(n-1)$;

(iii) In some cases the space dimension does not change. Thus, for the second branch of Theorem 2.3 we have

$$\bar{n}_2 - 1 = \frac{2m(n-1)}{n-2-mn}. \tag{2.42}$$

In particular $n = 1$ implies $\bar{n}_2 = 1$, this is a case in which the transformation does not imply a change of dimension. For the other branch we have

$$\bar{n}_1 - n = \frac{(n+2)(m_s - m)}{2m}, \tag{2.43}$$

so that dimension is preserved for $m = m_s$.

2.5 Improved correspondence for self-similar solutions

The correspondence relations (2.2) and (2.4) have the following disadvantage in the applications: in order to obtain an explicit solution of the PLE we have to integrate a solution of the PME multiplied by a weight, and this is not always easy. In the particular case of the self-similar solutions, we will obtain other relations, expressing directly $\bar{f}$ as a function of f and f'. We exhibit all the correspondence relations between self-similar profiles in the next two statements, where D_1 and D_2 are the same as in Theorems 2.1 and 2.2, and the dependence of $\bar{\alpha}$ on α, m and n is given in Theorem 2.3.

Proposition 2.1. *Suppose that* $0 < n < 2$ *and* $\bar{\alpha} \neq 0$. *Then the PLE and the PME profiles satisfy the following equalities:*

$$\bar{f}'(\bar{\eta}) = D_1 \eta^{\frac{2n-2}{m+1}} f(\eta) \tag{2.44}$$

and

$$\bar{f}(\bar{\eta}) = -\frac{mn-n+2}{m+1}\eta^{n-1}\frac{D_1}{\bar{\alpha}}(|f(\eta)|^{m-1}f'(\eta) + \beta\eta f(\eta)), \tag{2.45}$$

where $\bar{\eta} = \eta^{(mn-n+2)/(m+1)}$, *with the convention on* ε *made in the introduction.*

Proposition 2.2. *Suppose that* $2 < n < \infty$ *and* $\bar{\alpha} \neq 0$. *Then the PLE and the PME profiles satisfy the following equalities:*

$$\bar{f}'(\bar{\eta}) = D_2 \eta^{\frac{2}{m+1}} f(\eta) \tag{2.46}$$

and

$$\bar{f}(\bar{\eta}) = -\frac{2m}{m+1}\frac{D_2}{\bar{\alpha}}(\eta|f(\eta)|^{m-1}f'(\eta) + \beta\eta^2 f(\eta) + \frac{n-2}{m}|f(\eta)|^{m-1}f(\eta)), \tag{2.47}$$

where $\bar{\eta} = \eta^{(2m)/(m+1)}$.

Proof. We sketch the proofs of both proposition together, since the calculations are similar. First, we remark that (2.44), (2.46) and the correspondence between $\bar{\eta}$ and η are particular cases of (2.2), (2.4), resp. the relations between $\bar{r}$ and r. Indeed, we substitute in (2.2) u and $\bar{u}$ by their self-similar forms and we obtain:

$$t^{-\bar{\alpha}-\bar{\beta}}\bar{f}'(\bar{\eta}) = D_1 t^{\frac{2(n-1)}{m+1}\beta-\alpha}\eta^{\frac{2(n-1)}{m+1}} f(\eta)$$

Using the relations between the exponents given in Theorem 2.3 and the fact that $(m-1)\alpha+2\beta=\varepsilon$, we see that the exponents of the part with time are the same in both sides, hence we obtain (2.44). A similar calculation shows that the same happens in the case $n>2$ and for the other two types of self-similarity. We omit the details, since the calculations are straightforward.

In order to derive the second relation, we introduce (2.44) in the first case, respectively (2.46) in the second case, into the profile equation of the PLE (2.30). Suppose that $0<n<2$. Then we have

$$\bar{\alpha}\bar{f}(\bar{\eta})=-D_1^m\eta^{n-1}\frac{m(m+1)}{mn-n+2}|f|^{m-1}f'-D_1\bar{\beta}\eta^n f(\eta),$$

which transforms easily into (2.45) using the correspondence between exponents. In the same way we obtain (2.47), we omit again the details. $\square$

Remarks: (i) Due to the invariance of both equations under change of signs, we can also accept the opposite sign in (2.44) and (2.46). We will use both variants in the sequel, without specifying it, assuming in these transformations the sign which seems to be more convenient.
(ii) We can arrive to the same relations (2.44), (2.45), (2.46) and (2.47), by equating

$$\bar{\Phi}(\bar{\eta})=\Phi(\eta),\quad \bar{\Psi}(\bar{\eta})=\Psi(\eta),$$

in the phase-plane variables, using the definitions (2.27) in the PME case and (2.36) in the PLE case. The calculations are larger in this way, but end with the same results.

The limit case $\bar{\alpha}=0$. We remark that (2.45) and (2.47) are valid for all the exponents $\bar{\alpha}\neq 0$. The limit case $\bar{\alpha}=0$ is interesting because it corresponds to solutions which keep the same "vertical size" in time. It has to be investigated separately. We use directly the profile equation, which becomes

$$\bar{\eta}^{1-\bar{n}}((|\bar{f}'(\bar{\eta})|)^{p-2}\bar{f}'(\bar{\eta})\bar{\eta}^{\bar{n}-1})'+\frac{1}{p}\bar{\eta}\bar{f}'(\bar{\eta})=0. \tag{2.48}$$

We put $U(\bar{\eta})=\bar{f}'(\bar{\eta})$ and we make the change of variables $Z(\bar{\eta})=U(\bar{\eta})^{p-1}\bar{\eta}^{\bar{n}-1}$. The equation of Z writes:

$$Z'(\bar{\eta})+\frac{1}{p}\bar{\eta}^{\bar{n}-(\bar{n}-1)/(p-1)}Z(\bar{\eta})^{1/(p-1)}=0. \tag{2.49}$$

This equation is very easy to integrate and we find

$$U(\bar{\eta})=\bar{\eta}^{(1-\bar{n})/(p-1)}\left(C-\frac{p-2}{p(\bar{n}(p-2)+p)}\bar{\eta}^{(\bar{n}(p-2)+p)/(p-1)}\right)_+^{1/(p-2)}. \tag{2.50}$$

after coming back to the initial variables. This is the derivative of our solution, which exists for $p\neq p_c$. We will denote this special profile by F. It is easy to see, from the explicit expression of the derivative, that

$$\bar{f}(\bar{\eta})\sim\bar{\eta}^{(p-\bar{n})/(p-1)}$$

near $\bar{\eta} = 0$, hence $\lim_{\bar{\eta}\to 0} F(\bar{\eta}) = 0$ for $\bar{n} < p$ and it becomes singular at $\bar{\eta} = 0$ for $\bar{n} > p$. If $p > p_c$, this solution has a curious character, since it has no free boundary in the classical sense, but it stabilizes to a constant at the point where its derivative has a free boundary. This free boundary disappears for $p < p_c$. This solution has also been obtained as an example by Bidaut-Véron [27]. On the other hand, a similar solution with $\bar{\alpha} = 0$, but with $p < 1$, is important in image contour enhancement, as proposed by Barenblatt and Vázquez [14]. This case will be treated in detail in the second part of the chapter.

2.6 The slow-diffusion case

In this section we treat in more detail the case $m > 1$, resp. $p > 2$ and we obtain new self-similar solutions of the PLE through the correspondence relations we have established.

2.6.1 Changing sign self-similarity solutions of Type I

The relations between profiles will give us characterizations of the compactly supported self-similar solutions of Type I of the PLE, since the PME case is already well understood. For $0 < n < 2$, a full analysis of self-similarity has been done by Hulshof in [75]. In fact, the paper presents the complete classification only for $n = 1$, but the same analysis can be done under conditions of radial symmetry for all $n \in (0, 2)$, without essential changes. In particular, in these dimensions we have two explicit profiles, which are the Barenblatt profile, given by (2.6), and the dipole profile, given by (2.10). Moreover, there is a sequence of exponents

$$\alpha_1 = \frac{n}{n(m-1)+2} < \alpha_2 = \frac{1}{m} < \alpha_3 < ... \uparrow \frac{1}{m-1}, \tag{2.51}$$

such that there exist self-similar solutions with compact support if and only if $\alpha = \alpha_k$ for some positive integer k (see [75] and [76]). The sequence α_k will be called in the sequel the *Hulshof sequence of exponents* for the PME. In order to make an easier identification, we will call the sequence α_{2k} as the *dipole sequence* and the sequence α_{2k+1} as the *Barenblatt sequence*, after the name of the distinguished representative of each class. On the other hand, the classification of self-similar solutions of the PME in dimension $n > 2$ is presented in [76] and it is very similar to the one above, but the solutions corresponding to α with even index become singular at $\eta = 0$ (see for example the explicit dipole solution (2.10)) and they are not considered as solutions of the PME in the whole space. But we will still keep them in our analysis, though the functions $u(x,t)$ they generate are singular at $x = 0$. We will say, accepting an abuse of language, that a profile f is *odd* if $f(0) = 0$, $f^{'}(0) \neq 0$, and similarly for $\bar{f}$, and that f is *even* if $f(0) > 0$ and $f^{'}(0) = 0$.

Starting from this classification, we will obtain a new Hulshof-type sequence of exponents and solutions for the PLE. A difference will appear between the cases $0 < \bar{n} < p$ and $\bar{n} \geq p$. The main result of this subsection is:

Theorem 2.4. *Suppose $p > 2$.*

(a) For any $\bar{n} \in (0, p)$, there exists a sequence of exponents

$$\bar{\alpha}_1 = \frac{\bar{n}}{\bar{n}(p-2)+p} < \bar{\alpha}_2 < \bar{\alpha}_3 < \ldots \uparrow \frac{1}{p-2} \tag{2.52}$$

such that the PLE in dimension $\bar{n}$ has a compactly supported self-similar solution of Type I if and only if $\bar{\alpha} = \bar{\alpha}_k$ for some positive integer k. Moreover, the solution corresponding to the exponent $\bar{\alpha}_k$ has exactly $k-1$ changes of sign. The profiles of solutions with $\bar{\alpha} = \bar{\alpha}_k$ are even for k odd and odd for k even. The solution with $\bar{\alpha} = \bar{\alpha}_2$ plays the role of the dipole solution of the PME (but there is no simple algebraic expression for it).

(b) For any $\bar{n} \in [p, \infty)$, the first of the above mentioned classes become singular solutions at $\eta = 0$, while the odd terms α_{2k+1} correspond to even profiles with no singularity.

Proof. (a) Let us start with dimension $0 < n < 2$ in the porous medium case. We translate the mentioned results from [75] and [76] in terms of the p-Laplace equation, using the correspondences established before. Since we are considering only dimensions $0 < n < 2$, we are in the first case and we obtain all the values $\bar{n} \in (0, p)$. In particular, although in the porous medium case these dimensions have no physical sense except for $n = 1$, in the case of the p-Laplace equation we get many integer dimensions.

Using (2.45), we remark that, except for the Barenblatt profile, which gives $\bar{\alpha} = 0$, the other profiles obtained for $\alpha = \alpha_k$ with $k \geq 2$ in the porous medium case correspond to profiles with free boundary in the p-Laplace case. Analyzing the Barenblatt profiles, the corresponding ones in the p-Laplace case have the derivative

$$\bar{f}'(\bar{\eta}) = \bar{\eta}^{(1-\bar{n})/(p-1)} \left(C - \frac{p-2}{p(\bar{n}(p-2)+p)} \bar{\eta}^{(\bar{n}(p-2)+p)/(p-1)} \right)_+^{1/(p-2)}$$

i. e., we find the special profile F. In the same way, by applying the correspondence (2.45) to the dipole solution of the porous medium equation in dimension $0 < n < 2$, we obtain the Barenblatt profile of the PLE. The correspondence implies a change of the free parameter; indeed, in order to obtain a Barenblatt profile of the PLE with constant $C > 0$, we have to start from a dipole solution of the PME with free parameter

$$C \Big(\frac{m+1}{mn-n+2} \Big)^{(m+1)/m} \frac{1}{(2m)^{1/m}}.$$

Let us examine the general case. Suppose that we start from the solution of the PME with $\alpha = \alpha_{2k+1}$, which has the profile f with $f(0) = C > 0$, it is even and has exactly $2k$ changes of sign. From (2.45) we deduce that the corresponding profile $\bar{f}$ in the p-Laplace case has free boundary and from (2.44) that it has $2k-1$ changes of sign. Hence, it is a profile satisfying $f(0) = 0$. On the other hand, starting with a profile of dipole type, i.e. with $\alpha = \alpha_{2k}$ in the PME case, we obtain a profile having free boundary and has $2k-2$ changes of sign, using (2.44).

We have to analyze the behavior at 0, using (2.45) and the local analysis results obtained in [75]. A profile f with $\alpha = \alpha_{2k}$ has the property that

$$\lim_{\eta \to 0} f(\eta) = 0, \quad \lim_{\eta \to 0} \eta^{n-1} (|f|^{m-1} f)'(\eta) = C > 0$$

as it results from Proposition 2.3 in [75] after performing straightforward changes. From (2.45) we obtain that $\lim_{\bar{\eta}\to 0} \bar{f}(\bar{\eta}) = C_1 > 0$, hence these are not odd profiles. In the other case, a profile with $\alpha = \alpha_{2k+1}$ satisfies $f(0) = C > 0$, $f'(0) = 0$. For $0 < n < 1$, it appears a singularity in (2.44) at 0, given by $\eta^{(2n-2)/(m+1)}$. But for any $n \in (0,1)$, this singularity is integrable, and $\bar{f}(0) = 0$. This shows that for $\bar{n} \in (0,p)$, the dipole-type solutions are real solutions, i.e. non-singular in the origin.

If we define

$$\bar{\alpha}_{2k} = \frac{(mn - n + 2)\,\alpha_{2k+1} - n}{2}, \tag{2.53}$$

we obtain the desired sequence of exponents in (2.52). The description above, together with the uniqueness in the case of the PME, implies that these are all the profiles obtained from the PME in the case $0 < n < 2$. Moreover, the sequence $(\bar{\alpha}_k)_{k\geq 1}$ is increasing. Since $\alpha_k \to \frac{1}{m-1}$ in the PME case, (2.53) implies that $\lim_{k\to\infty} \bar{\alpha}_k = \frac{1}{p-2}$.

On the other hand, we can obtain all the dimensions in the range $\bar{n} \in (0,p)$ also from the dimensions $2 < n < 2(m+1)$ in the PME case, using in this case the other correspondence for the dimensions. In this range, the corresponding formulas for the profiles are (2.46) and (2.47). This implies a correspondence between the Barenblatt profiles of the PLE and PME at a qualitative level, but again with a change of free parameter; indeed, in order to obtain a Barenblatt profile of the PLE with constant $C > 0$, we have to start with a Barenblatt profile of the PME with free parameter

$$C\beta^{-1/m}\frac{m+1}{2(n(m-1)+2)}\Big(\frac{m+1}{2m}\Big)^{1/m}. \tag{2.54}$$

If we start with the profile having $\alpha = \alpha_{2k+1}$ in the PME case, which is even and has $f(0) = C > 0$ and $f'(0) = 0$, using (2.47), we deduce that

$$\bar{f}(0) = \frac{2(n-2)}{m(m+1)}\frac{D}{\bar{\alpha}}C^m > 0,$$

hence it is also an even profile. This is due to the appearance of the term $((n-2)/m)|f|^{m-1}f$ in the formula for this range of dimensions. Here $\bar{\alpha} = m\alpha - 1$. On the other hand, starting from the profile in (2.10), we obtain in this case the special profile F with $\bar{\alpha} = 0$. This is not only a coincidence.

This is a particular effect of the self-map of the PME described in Section 2.3. We remark that in $n = 2$ we have a coincidence of the exponents: $\alpha_{2k} = \alpha_{2k-1}$ and of the corresponding profiles of the PME (in particular the dipole and the Barenblatt profiles coincide, see (2.6) and (2.10)). For $n > 2$, the dipole series passes above the Barenblatt series; at the same time the dipole series does not produce real solutions anymore (the profiles become singular at 0), but it still exist. In general, if we fix $\bar{\eta}$ and the PLE profile $\bar{f}$ and we denote by η_i and f_i the independent variable and the PME profile in dimension n_i, $i = 1, 2$, where $0 < n_1 < 2$, $2 < n_2 < 2(m+1)$, we obtain:

$$\eta_2 = \eta_1^{\frac{mn_1 - n_1 + 2}{2m}}$$

and

$$f_2(\eta_2) = \frac{D_1}{D_2}\eta_1^{\frac{2n_1-2}{m+1}}\eta_2^{-\frac{2}{m+1}}f_1(\eta_1) = \frac{D_1}{D_2}\eta_1^{\frac{n_1-2}{m}}f_1(\eta_1).$$

Since the solutions in the dipole series of the PME behave near $\eta = 0$ like $\eta^{(2-n)/m}$, the above calculation exemplifies the self-map of the PME: the Barenblatt series in dimension n_2 with $2 < n_2 < 2(m+1)$ corresponds by this self-map to the dipole series in dimension $n_1 = \frac{2(2m+1-n_2)}{mn_2-n_2+1} < 2$, and the (virtual) dipole series in dimension n_2 corresponds to the Barenblatt series in dimension n_1.

We also remark that through the transformations of dimensions, from the dimensions n_1 and n_2 we obtain the same dimension $\bar{n}$ in the PLE case. Hence the two-sided correspondence gives nothing new: the dipole series in the PLE comes from the Barenblatt series of the PME in dimension $0 < n_1 < 2$ (except from the first representant, giving the special profile F) and from the dipole series of the PME in dimension n_2. The Barenblatt series of the PLE comes from the dipole series of the PME in dimension n_1 and from the Barenblatt series of the PLE in dimension n_2. Using the formulas (2.45) and (2.47) and the fact that the dipole profiles in the PME case behave like $\eta^{(2-n)/m}$ near 0, it is easy to remark that the dipole series contains real (i.e. nonsingular at 0) solutions for $0 < \bar{n} < p$ and singular solutions for $\bar{n} \geq p$.

We still have to prove the uniqueness of such solutions. Suppose that we have a self-similar solution of the PLE with another exponent $\bar{\alpha} \neq \bar{\alpha}_k$. Then, by reversing the transformations we have done, we obtain a solution of the PME with the profile

$$f(\eta) = -\frac{1}{D_1}\bar{f}'(\bar{\eta})\eta^{\frac{2-2n}{m+1}} \tag{2.55}$$

in the case $n < 2$, or

$$f(\eta) = -\frac{1}{D_2}\bar{f}'(\bar{\eta})\eta^{-\frac{2}{m+1}} \tag{2.56}$$

if $n > 2$. Since, from (2.45) and (2.47), the flux condition $(f^m)' = 0$ at the free-boundary point is accomplished, this profile comes from a compactly supported self-similar solution of the PME having another exponent than those in the sequence $(\alpha_k)_{k\geq 1}$. But this is a contradiction with the uniqueness in the PME case, proved in [75] and [76]. This ends the proof of part (a).

(b) This is now easy, taking into account the analysis made before. If $\bar{n} \geq p$, it may come only from $n \geq 2(m+1)$ in the PME. In this case, the Barenblatt series of the PME transforms into the Barenblatt series of the PLE, as before, and the dipole series of the PME transforms into the dipole series of the PLE. But we remark that for $n > 2(m+1)$, the term $\beta\eta^2|f|$ from equation (2.47) behaves near 0 like $\eta^{(2(m+1)-n)/m}$, hence it becomes singular. The only real solutions are that of the Barenblatt series in this case. The uniqueness part is treated in the same way as in part (a). $\square$

As an interesting application, the dipole solutions of the PLE introduced above represent the asymptotic profiles of the general solutions of the PLE in a domain with holes in dimensions $\bar{n} < p$, see [81].

We indicate in the following tables how the Barenblatt solutions of the two equations change. The first table contains the solutions of the PLE corresponding to the Barenblatt solution of the PME in dimension n. As we state in the introduction, we do not consider the case $n = 2$, $\bar{n} = 0$.

n	$\bar{n}$	Solution PLE
$0 < n < 2$	$0 < \bar{n} < p$	Special solution F with $\bar{\alpha} = 0$
$2 < n < \infty$	$0 < \bar{n} < \infty$	Barenblatt PLE

Table 2.1: Correspondence Barenblatt PME $\longrightarrow$ solutions PLE

The next table contains the solutions of the PME corresponding to the Barenblatt solution of the PLE in dimension $\bar{n}$.

$\bar{n}$	n	Solution PME
$\bar{n} \geq p$	$n \geq 2(m+1)$	Barenblatt PME
$0 < \bar{n} < p$	$0 < n_1 < 2$	Dipole PME
$0 < \bar{n} < p$	$0 < n_2 < 2(m+1)$	Barenblatt PME

Table 2.2: Correspondence Barenblatt PLE $\longrightarrow$ solutions PME

Nonlinear eigenvalue sequence

Similarly to the PME case [24], we define the eigenvalues of the PLE:

$$\bar{k}_j = \frac{\bar{\alpha}_j}{\bar{\beta}_j}. \tag{2.57}$$

Since $(p-2)\bar{\alpha}_j + p\bar{\beta}_j = 1$, it follows that

$$\bar{\beta}_j = \frac{1}{(p-2)\bar{k}_j + p}, \quad \bar{\alpha}_j = \frac{\bar{k}_j}{(p-2)\bar{k}_j + p}. \tag{2.58}$$

Thus, we can establish a correspondence between the eigenvalues of the PME and of the PLE. For $0 < n < 2$, we have

$$\bar{k}_j = \frac{(m+1)((mn-n+2)\alpha_j - n)}{2(mn-n+2)\beta_j}$$

and we have to take into account that

$$\alpha_j = \frac{k_j}{k_j(m-1)+2}, \quad \beta_j = \frac{1}{(m-1)k_j+2}.$$

In the case $n > 2$, the correspondence is

$$\bar{k}_j = \frac{(m\alpha_j - 1)(m+1)}{2m\beta_j}.$$

We replace in the expressions of $\bar{k}_j$ the values of α_j and β_j and we obtain

$$\bar{k}_j = \frac{(k_j - n)(m+1)}{mn - n + 2}, \quad \bar{k}_j = \frac{(k_j - 2)(m+1)}{2m} \tag{2.59}$$

in the cases $0 < n < 2$ and $n > 2$ respectively. We remark that in the first case, from $k_1 = n$ we obtain $\bar{k}_1 = 0$, corresponding to the solution with the special profile F. From $k_2 = 2$ (which corresponds to the dipole case in the PME), we obtain $\bar{k}_2 = \bar{n}$, which is the eigenvalue for the Barenblatt orbit in the PLE. In the case $n > 2$, from $k_1 = n$ we obtain $\bar{k}_1 = \bar{n}$, which also confirms the correspondence of solutions established in the proof of Theorem 2.4.

Remarks: (i) In dimension $n = 1$ (and consequently $\bar{n} = 1$), we have a very simple correspondence: $\bar{k}_j = k_j - 1$. In particular $\bar{k}_3$ has the following properties: $\bar{k}_3 > 2$ and $\bar{k}_3 \to 3$ as $p \to \infty$. This follows from the results about k_3 in [24].
(ii) For any eigenvalue k_j, there exists an entire orbit of self-similar solutions of the PME, which can be obtained from a particular one by rescaling: $f_\lambda(\eta) = \lambda^{-2} f(\lambda^{m-1}\eta)$. The same is valid in the PLE case: it is easy to see that the rescaling

$$\bar{f}_\lambda(\bar{\eta}) = \lambda^{-p} \bar{f}(\lambda^{p-2}(\bar{\eta}))$$

produces self-similar profiles with the same eigenvalue (in fact with the same exponents). The equivalence of the phase-planes grants the uniqueness (i.e. the fact that there are no other profiles in the orbit).

2.6.2 Self-similar solutions of Type II

We are concerned in this part with the focusing self-similar solutions, studied by Aronson and Graveleau [7] for the PME case and in Gil and Vázquez [65] for the PLE. In [7], the authors prove that there exists a family of self-similar solutions of the focusing problem for the PME. These are solutions of Type II which are positive in an interval $(r(t), \infty)$ with $r(t) \to 0$ in some finite time $T > 0$. These solutions exist for some particular value of the exponent β (that we denote by β_0) such that

$$1 + \frac{(m-1)(n-2)}{(m-1)(n+2)+4} < -\frac{1}{\beta_0} < 2.$$

We want to translate this into the PLE case with $p = m + 1$. It suffices to consider the case $n > 2$ in the PME which covers all possible $\bar{n} > 0$. Since $-1/\beta_0 < 2$, using (2.41) it follows that

$$-\frac{1}{\bar{\beta}_0} = -\frac{m+1}{2m\beta_0} < \frac{p}{p-1}. \tag{2.60}$$

On the other hand, using (2.40) and (2.41), we obtain in the PLE case that

$$1 + \frac{(p-2)(\bar{n}-p)_+}{(p-1)(\bar{n}(p-2)+2p)} \leq -\frac{1}{\bar{\beta}_0} \tag{2.61}$$

The estimates (2.60) and (2.61) together form precisely the result obtained in [65] for the focusing self-similar solutions of the PLE. The correspondence of the profiles is the same as above.

2.7 Two limit cases: heat equation and eikonal equation

In this section we deal with the limit cases obtained when $m \to 1$ and $p \to 2$. Depending on the manner in which we pass to the limit, there are two possible limit equations: the heat equation and the eikonal equation. Many of the calculations will be formal, but they are rigourously true in the explicit cases.

7.1. The heat equation as a limit case. The general transformations were already presented at the end of Section 2.3. In dimension $n > 2$, by particularizing the general relations to self-similar solutions (and choosing the minus sign for convenience), we obtain:

$$\bar{f}'(\bar{\eta}) = -\eta f(\eta), \quad \bar{\eta} = \eta, \tag{2.62}$$

which is the usual equation satisfied by the exponential profiles which are solutions of the heat equation. This is true for example if one starts with the Barenblatt profile (2.8) with $C = 1$ and correspondingly with the profile (2.6) with the free parameter C given by (2.54) and pass to the limit. With these constants, the limit process can be done and we obtain from the PLE solution the well-known Gaussian profile

$$f(\eta) = \bar{f}(\bar{\eta}) = e^{-\frac{\eta^2}{4}}$$

and from the corresponding PME profile the same Gaussian profile, but divided by 2. The equality (2.62) is obviously verified in this case. We can say, together with the equality of the coefficients in Section 2.4 and with the convention that we can set multiplicative constants to 1 (since the equation is linear), that in the limit we obtain the identity map in dimension $n > 2$.

In dimension $n < 2$, by translating again the general radial correspondences in terms of self-similar profiles, we have:

$$\bar{f}(\bar{\eta})' = -f(\eta)\eta^{n-1}, \quad \bar{\eta} = \eta, \tag{2.63}$$

which implies in the limit that the Gaussian profile corresponds to another solution. For example, in dimension $n = 1$, passing to the limit in (2.10) with $C = 1$, we obtain a dipole-type profile for the heat equation:

$$f(\eta) = K\eta e^{-\frac{\eta^2}{4}}, \tag{2.64}$$

which corresponds to the Gaussian map through (2.63). Hence in this case, the limit transform is not an identity, as expected, but it transforms solutions of the heat equation into different solutions of the heat equation.

7.2. The Eikonal equation as a limit case. This comes from a different way of passing to the limit, that we will describe in the next lines. We pass to the so-called pressure variables in both equations (see [129]):

$$v(x,t) = \frac{1}{m-1}|u(x,t)|^{m-1}, \quad \bar{v}(x,t) = \frac{p-1}{p-2}|\bar{u}(x,t)|^{\frac{p-2}{p-1}} \tag{2.65}$$

and the equations that satisfy v and $\bar{v}$ are

$$v_t = (m-1)v\Delta v + |\nabla v|^2, \quad \bar{v}_t = \frac{p-2}{p-1}\bar{v}\Delta_p \bar{v} + |\nabla \bar{v}|^p. \tag{2.66}$$

We can pass to the limit in both equations and obtain the same limit equation

$$v_t = |\nabla v|^2, \tag{2.67}$$

which is the *eikonal equation*, arising in geometrical optics. The limit process is presented rigourously by Aronson and Vázquez [9], see also [102]. We pass to radial variables and obtain the explicit correspondence in the limit. All the calculations below will be formal in general and will be rigourously true in the regions where u, $\bar{u} \neq 0$. For $n < 2$, from (2.2), we have:

$$\frac{|\bar{u}^{'}|^{m-1}}{m-1} = \frac{(mn-n+2)^2}{m(m+1)^2} r^{\frac{2(n-1)(m-1)}{m+1}} \frac{|u|^{m-1}}{m-1}. \tag{2.68}$$

Since $\bar{v} = \frac{p-1}{p-2}|\bar{u}|^{\frac{p-2}{p-1}}$, it follows that

$$\frac{|\bar{u}^{'}|^{p-2}}{p-2} = \frac{|\bar{v}^{'}|^{p-2}}{p-2}|\bar{u}|^{\frac{p-2}{p-1}} = \frac{|\bar{v}^{'}|^{p-2}}{p-1}\bar{v}. \tag{2.69}$$

Taking into account that $p = m+1$ and letting $m \to 1$ and $p \to 2$ in (2.68) and (2.69), we obtain in the (formal) limit that $\bar{v} = v$. For $n > 2$, we perform a similar calculation, starting from (2.4), which becomes

$$\frac{|\bar{u}^{'}|^{m-1}}{m-1} = \frac{(2m)^2}{m(m+1)^2} r^{\frac{2(m-1)}{m+1}} \frac{|u|^{m-1}}{m-1}. \tag{2.70}$$

Using again (2.69), taking into account that $p = m+1$ and letting $m \to 1$ and $p \to 2$ in (2.69) and (2.70), we obtain in the (formal) limit that $\bar{v} = v$. Hence the correspondence of PLE and PME transforms into identity at this formal level, in the whole range of dimensions.

We study the effect of our correspondence relations on self-similar solutions. If we pass to the limit as $m \to 1$ in the pressure functions obtained from the Barenblatt solution (2.6) and the dipole (2.10), we obtain the same explicit family of Oleinik profiles $(C - |x|^2/4t)_+$ of the eikonal equation. The same happens with the two explicit profiles we have for the PLE, i.e. the special profile F with $F^{'} = U$ in (2.50) and the Barenblatt solution (2.8).

If we look for self-similar solutions of type I for the eikonal equation (where we put g instead of f to avoid confusions), we obtain that the exponents satisfy $\alpha + 2\beta = 1$ and the profile equation is

$$g^{'}(\eta)^2 + \alpha g(\eta) + \beta \eta g^{'}(\eta) = 0. \tag{2.71}$$

Since $(m-1)\alpha + 2\beta = 1$ in the PME case and $(p-2)\bar{\alpha} + p\bar{\beta} = 1$ in the PLE case, passing to the limit as $m \to 1$ and $p \to 2$, we always obtain that $\beta, \bar{\beta} \to \frac{1}{2}$. Passing also to the limit in the expression of the pressure functions obtained from solutions of the form (2.11), we obtain that $\alpha, \bar{\alpha} \to 0$. By inserting this in (2.71) and integrating, we easily deduce that all the compactly supported self-similar solutions of type I of the PME and PLE converge to the same family of profiles $(C - |x|^2/4t)_+$, with $C > 0$. Summing up, the different behavior of the self-similar solutions in this limit case with respect to the previous one (the heat equation) comes from the fact that the critical powers $1/(m-1)$ and $(p-1)/(p-2)$ in the profiles disappear.

2.8 The fast-diffusion case

The important difference between the fast-diffusion case (i. e., $m < 1$ in the PME and $p < 2$ in the PLE) and the classical slow-diffusion case is that the degeneracy of the equation appears not for $u = 0$ (resp. $\nabla u = 0$), but as it goes to infinity, while at zero the equation becomes singular. The main consequence of this difference is that in the fast-diffusion case there are no nontrivial compactly supported self-similar solutions, and we have to classify solutions by their decay rate near infinity.

2.8.1 Self-similar solutions of Type I

We classify in this subsection the self-similar solutions of Type I for the PLE with $p_c < p < 2$. As we have already said, the relation between the exponents is

$$(1-m)\alpha + 1 = 2\beta, \quad (2-p)\bar{\alpha} + 1 = p\bar{\beta}. \tag{2.72}$$

We will ask, as usual, that $\alpha \geq 0$ and $\bar{\alpha} \geq 0$. Notation: we say that a profile $\bar{f}$ has a decay rate at infinity like $\bar{\eta}^{-l}$, where $l > 0$, if $\bar{\eta}^l \bar{f}(\bar{\eta}) \to C$ as $|\bar{\eta}| \to \infty$, where C denotes a nonzero constant. We will write $\bar{f} \approx \bar{\eta}^{-l}$.

Theorem 2.5. *The maximal possible decay rate of the profiles of self-similar solutions of the PLE is $\bar{\eta}^{p/(p-2)}$. We will say that a profile has optimal decay at infinity if it decays like this precise power of $\bar{\eta}$.*
(a) Suppose that $0 < \bar{n} < p$. Then there exists a sequence of numbers

$$0 = \bar{k}_0 < \bar{n} = \bar{k}_1 < \bar{k}_2 < \ldots < \frac{p}{2-p} \tag{2.73}$$

such that there exists a self-similar solution of the PLE whose profile has optimal decay at infinity in the above sense and such that it corresponds to a solution with $\bar{\alpha}/\bar{\beta} = k$ if and only if $\bar{k} = \bar{k}_j$ for some integer j. We will again call $\bar{k}_j$ the nonlinear eigenvalues. The profiles corresponding to eigenvalues with even index satisfy $f(0) = 0$ and the profiles corresponding to eigenvalues with odd index satisfy $f(0) > 0$, $f'(0) = 0$.
(b) Suppose $p \leq \bar{n} < \frac{p}{2-p}$. Then the result is similar as in part (a), except from the fact that there are no solutions with profiles satisfying $f(0) = 0$ and having optimal decay.
(c) If $\bar{n} \geq \frac{p}{2-p}$ there are no self-similar solutions of Type I with optimal decay.

Proof. We translate to PLE the similar result about the PME from [24]. There, the analysis is done only in the case $n = 1$, but it can be done with minor changes for $0 < n < 2/(1-m)$. In that paper it is proved that there exists a sequence of eigenvalues $(k_j)_{j\geq 1}$ such that there are profiles with optimal decay $2/(1-m)$ if and only if $k = k_j$, where $k = \alpha/\beta$.

Let us prove first that the optimal decay is $p/(2-p)$ for the PLE. In the fast diffusion case, we still can define the two explicit profiles of solutions of the PLE, the special profile F whose derivative has the expression (2.50), and the Barenblatt profile (2.8). From the explicit expression, it is obvious that the Barenblatt profile decays like $\bar{\eta}^{p/(p-2)}$. On the other hand, from (2.50), we remark that

$$F^{'}(\bar{\eta}) \approx \bar{\eta}^{2/(p-2)},$$

hence the special solution F decays also like $\bar{\eta}^{p/(p-2)}$.

Suppose there exists a self-similar solution of the PLE whose profile satisfies $\bar{f}(\bar{\eta}) \approx \bar{\eta}^{-l}$, with $l > p/(2-p)$. Then, from (2.44) or (2.46), we obtain that the corresponding profile in the PME case satisfies

$$f(\eta) = -C\eta^{-\frac{2n-2}{m+1}}\bar{f}^{'}(\bar{\eta}) \approx \eta^{-\frac{2n-2}{m+1}}\bar{\eta}^{-l-1} = \eta^{-\frac{2n-2}{m+1}+(-l-1)\frac{mn-n+2}{m+1}}$$

in the case $0 < n < 2$, or

$$f(\eta) = -C\eta^{-\frac{2}{m+1}}\bar{f}^{'}(\bar{\eta}) \approx \eta^{-\frac{2}{m+1}}\bar{\eta}^{-l-1} = \eta^{-\frac{2}{m+1}+(-l-1)\frac{2m}{m+1}}$$

in the case $n > 2$. Since $-l-1 < 2/(p-2) = 2/(m-1)$, we obtain that in both cases the profile $f(\eta)$ has a better decay than $\eta^{2/(m-1)}$, which is not possible, since it contradicts the result in [24]. Hence the decay like $\eta^{p/(p-2)}$ is optimal.

To prove part (a), we use the similar result in [24] and we translate it into the PLE. From the first part of the proof, we already know that optimal decay in the PME case transforms into optimal decay in the PLE case. The relation between the eigenvalues is again (2.59) and if we take a profile f with optimal decay in the PME case corresponding to the eigenvalue k_j, it changes into a profile $\bar{f}$ with optimal decay in the PLE case corresponding to $\bar{k}_{j-1}$. We omit the details, since there are very similar to those of the proof of Theorem 2.4. The uniqueness is also immediate, and part (b) follows from the same discussion about the existence of dipole as above. For part (c), it is enough to remark that the Barenblatt exponents become negative (hence all the exponents associated to the other solutions from our series), hence there is no solution in the sense we look for. □

Remark: Part (c) of Theorem 2.5 contains as a particular case the well-known fact that for $n > p/(2-p)$, i.e. $p < p_c = 2n/(n+1)$, there are no Barenblatt solutions. In fact, we have no Type I solutions with nonnegative exponents in this subcritical case.

2.8.2 Self-similar solutions of Type III

We insert here a short discussion on solutions of Type III, since it holds for the critical cases $m = m_c$ and $p = p_c$. As their general formula shows, these solutions are eternal, i.e., they live for $-\infty < t < \infty$ and having an exponential decay in time. There is an important explicit example of this type, appearing in the critical fast-diffusion case of the PME, i.e. for $m = m_c$ (see [127]). We set

$$u(x,t) = \frac{1}{(a|x|^2 + Ce^{2nat})^{n/2}},$$

with free parameters a, $C > 0$. It can be easily seen that this solution can be written as

$$u(x,t) = e^{-n\beta t} f(xe^{-\beta t}), \quad \beta = na, \quad f(\eta) = (C + a\eta^2)^{-n/2}. \tag{2.74}$$

Having the same ratio n between the exponents α and β and a similar profile, this solution can be considered as an extension of the Barenblatt solutions into the critical case $m = m_c$. Since we are in the critical case, necessarily $n > 2$. By applying the transforms in Theorem 2.2 to this solution, we obtain a corresponding solution in the PLE case for $p = p_c$, having the profile and exponents

$$\bar{f}(\bar{\eta}) = \left(C + a\bar{\eta}^{p/(p-1)}\right)^{-(\bar{n}-1)/2}, \quad \bar{\beta} = a\frac{\bar{n}^2 - 1}{\bar{n}} \tag{2.75}$$

and $\bar{\alpha} = \bar{n}\bar{\beta}$, following from the relations between the eigenvalues stated at the end of Section 2.6. In a more explicit form, we have:

$$\bar{u}(x,t) = \frac{1}{(a|x|^{p/(p-1)} + Ce^{2\bar{n}\bar{\beta}t/(\bar{n}-1)})^{(\bar{n}-1)/2}}.$$

In this way, we obtain a new self-similar solution of Type III in the critical case $p = p_c$ of the PLE, which can be also seen as an extension into this case of the PLE Barenblatt solutions.

2.8.3 Self-similar solutions of Type II

The existence of this type of solutions is a very investigated subject in the case of subcritical fast diffusion. The existence of this type of solutions with special properties was formally discussed by King in [94] and a rigorous analysis was performed by Peletier and Zhang in [108], and continued in [127]. They belong to the class containing so-called *anomalous exponents*, since they are not calculated a priori from dimensional considerations but are the result of a phase-plane analysis. For more details about anomalous exponents the reader is referred to [12] and [8].

In all this subsection, we will work only in the cases $m < m_c$ and $p < p_c$, and implicitly $n > 2$. The simplest solutions are those of separate variables. In the PME case, such solutions appear in the book [127], Chapter 5, and have the exact expression:

$$U(x,t;T) = c_m \left(\frac{T-t}{|x|^2}\right)^{1/(1-m)}, \quad c_m^{1-m} = 2\left(n - \frac{2}{1-m}\right). \tag{2.76}$$

with $T > 0$ arbitrary. It is easy to see that $c_m^{1-m} = k^{-1}$, where k is the constant appearing in the expression of the Barenblatt solution (2.6). This solution is the closest relative in the subcritical range of the Barenblatt solutions of the range $m > m_c$. Using the correspondence formulas, we derive new solutions of the PLE having a similar expression:

$$\bar{u}(\bar{r},t) = c_p(T-t)^{\frac{1}{2-p}}\left(\frac{1}{\bar{r}^{p/(p-1)}}\right)^{\frac{p-1}{2-p}}, \quad c_p^{\frac{2-p}{p-1}} = \frac{p}{2-p}(-\bar{n}(p-2)-p)^{\frac{1}{p-1}}. \tag{2.77}$$

We remark that, similarly, $c_p^{(2-p)/(p-1)} = k^{-1}$, with k given in (2.9); we can also say that this solution is the closest relative of the Barenblatt solutions of the PLE. In both the PME and the PLE case, these solutions are singular at $x = 0$.

There exist other interesting self-similar solutions of the PLE in the range $p < p_c$. Using the results in the PME case and our correspondence relations, we prove:

Theorem 2.6. *For any $p \in (1, p_c)$, there is a unique $\bar{\gamma} = \bar{\gamma}(p)$ and a unique (up to a rescaling) self-similar solution of the form*

$$\bar{u}(x,t) = (T-t)^{\bar{\alpha}_0}\bar{f}(x(T-t)^{\bar{\beta}_0}) \tag{2.78}$$

with

$$\bar{\alpha}_0 = \frac{\bar{\gamma}}{\bar{\gamma}(2-p)-p}, \quad \bar{\beta}_0 = \frac{1}{\bar{\gamma}(2-p)-p}, \tag{2.79}$$

that we will call as anomalous exponents *in the sequel, such that the profile $\bar{f}$ is bounded and has a decay at infinity like $\bar{\eta}^{-(\bar{n}-p)/(p-1)}$.*

Proof. We start from the similar result about the existence of the anomalous exponents in the PME case, see [127], Theorem 7.1, or [108]. We use the relations between exponents and profiles. Since $m < m_c$ in the PME case, we are only in dimension $n > 2$. In the PME case, it is well-known that there exists a special Type II solution with anomalous exponents α_0 and β_0 and with decay at infinity like $\eta^{-(n-2)/m}$. Let f be the profile of this solution.

By (2.46), there exists a corresponding profile in the PLE case, given by

$$\bar{f}' = -D_2\eta^{\frac{2}{m+1}}f(\eta) \approx \bar{\eta}^{\frac{1}{m}}\eta^{-\frac{n-2}{m}} = \bar{\eta}^{\frac{1}{p-1}-\frac{\bar{n}}{m}} = \bar{\eta}^{-\frac{\bar{n}-1}{p-1}},$$

hence the decay of $\bar{f}$ at infinity is like $\bar{\eta}^{(p-\bar{n})/(p-1)}$. Since $m < m_c$, it follows that $n > 2/(1-m)$, hence $\bar{n} > m+1 = p$. Then the rate we have obtained is negative and represents a real decay, as desired. From the relations between exponents, we have:

$$\begin{aligned} \bar{\beta}_0(p) &= \frac{2m}{m+1}\beta_0(m) = \frac{2(p-1)}{p}\beta_0(p-1), \\ \bar{\alpha}_0(p) &= \frac{1+2m\beta_0(m)}{1-m} = \frac{1+2(p-1)\beta_0(p-1)}{2-p}, \end{aligned} \tag{2.80}$$

where we have emphasized the dependence of α_0 and β_0 on m or p. For every $p \in (1, p_c)$, from the correspondences we deduce that there exists a $m \in (0, m_c)$ corresponding to it. In this way we cover all the range of possible values of p. Finally, the uniqueness follows in a standard way from that of the special solution in the PME case. □

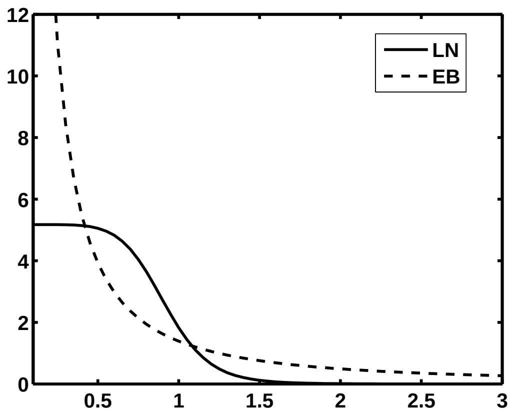

Figure 2.2: The two self-similar solutions of the PLE for $p = p_s$ and $n = 3$.

Let us now discuss in more detail the Yamabe case, i.e. $m = m_s$ in the PME case and $p = p_s$ in the PLE case. It is easy to see that in this case we have $\bar{n} = n$ and it is the unique fixed point of the correspondence of dimensions for $n > 2$. In this case, it is well-known (see [127]) that $\beta_0 = 0$ and $\alpha_0 = 1/(1-m) = (n+2)/4$. This permits an easy integration of the equation of the profiles and obtain the explicit Loewner-Nirenberg profile:

$$f(\eta; C) = \left(C + \frac{\eta^2}{4nC}\right)^{-\frac{n+2}{2}}, \tag{2.81}$$

which has been discovered in [103] and plays an important role in differential geometry. Using (2.47) and the correspondence of exponents, we obtain a new explicit profile for the PLE with $p = p_s$, in dimension n (the same):

$$\bar{f}(\bar{\eta}; C) = \frac{4n}{n+2}\left(\frac{4n^3C}{n^2-4}\right)^{\frac{n-2}{4}}\left(1 + C\bar{\eta}^{\frac{2n}{n-2}}\right)^{-\frac{n}{2}}, \tag{2.82}$$

having as exponents $\bar{\beta}_0 = 0$, $\bar{\alpha}_0 = 1/(2-p)$ and $C > 0$ arbitrary. On the other hand, in this case we still have the separate variable solutions indicated in (2.77), whose profile particularizes into

$$\bar{f}(\bar{\eta}) = \frac{2n^{n/2}}{(n+2)^{(n+2)/4}}\bar{\eta}^{-n/2}. \tag{2.83}$$

We remark that, as in the PME case analyzed in [127], the Loewner-Nirenberg profile has a better decay rate than the singular solution (2.83). We illustrate in Figure 2.2 the profiles of two solutions of the PLE in the case $p = p_s$, where the abbreviation LN indicates the profile in (2.82) and EB indicates the one from (2.83).

We end this section with some properties of the anomalous exponents $\bar{\alpha}_0(p)$ and $\bar{\beta}_0(p)$.

Theorem 2.7. *The anomalous exponents* $\bar{\alpha}_0$ *and* $\bar{\beta}_0$ *are both analytic functions of* p. *We have*

$$\lim_{p\to 1}\bar{\alpha}_0 = 1, \quad \lim_{p\to 1}\bar{\beta}_0 = 0, \quad \lim_{p\to p_c}\bar{\alpha}_0 = \infty, \quad \lim_{p\to p_c}\bar{\beta}_0 = \infty \tag{2.84}$$

Moreover, $\bar{\alpha}_0$ *is an increasing function of* p, *but* $\bar{\beta}_0$ *is not increasing.*

Proof. The analiticity follows easily from Theorem 7.2 in [127] and (2.80). To obtain the limits as $p\to 1$, we use an estimate from [27], stating that the following inequality holds:

$$\delta+\frac{\delta(\bar{n}-\delta)}{(p-1)(2\delta-\bar{n})}<\frac{\bar{\alpha}_0}{\bar{\beta}_0}<\min\left(-\frac{p}{p-1},\delta+\frac{\delta(\bar{n}-\delta)}{(p-1)(2\delta-\bar{n}-2\sqrt{(\bar{n}-\delta)p/(p-1)})}\right)<0,$$

where $p<p_s$ and $\delta=p/(2-p)$. Passing to the limit in this inequality, we obtain that for p very close to 1 holds true:

$$-\frac{K_1(p)}{p-1}\leq\frac{\bar{\alpha}_0}{\bar{\beta}_0}\leq-\frac{K_2(p)}{p-1}, \tag{2.85}$$

where

$$\lim_{p\to 1}K_1(p)=\frac{\bar{n}-1}{\bar{n}-2}, \quad \lim_{p\to 1}K_2(p)=1,$$

and we deduce that the anomalous eigenvalue $\bar{\gamma}_0=\bar{\alpha}_0/\bar{\beta}_0$ tends to $-\infty$ as $p\to 1$. On the other hand, using the relation between the exponents $(2-p)\bar{\alpha}_0=1+p\bar{\beta}_0$, we obtain easily from (2.85) that $\bar{\beta}_0\to 0$ as $p\to 1$. The limit of $\bar{\alpha}_0$ is now trivial.

Since $p<p_c$, it corresponds $m<m_c$, hence we are only in dimensions $n>2$ in the PME case. The relation between exponents is in this case:

$$\bar{\alpha}_0=m\alpha_0+1, \quad \bar{\beta}_0=\frac{2m}{m+1}\beta_0, \tag{2.86}$$

independently on the value of n. From the results in the PME case (Theorem 7.2 in [127]), we obtain the infinite limits of $\bar{\alpha}_0$ and $\bar{\beta}_0$, as stated. Moreover, since $\alpha_0>0$ and it is an increasing function of m, it follows that $\bar{\alpha}_0$ is an increasing function of p. On the other hand, there is no global monotonicity of $\bar{\beta}_0$, since we have $\bar{\beta}_0=0$ in $p=p_s$ (which corresponds to the Yamabe case presented above) and $\lim_{p\to 1}\bar{\beta}_0=0$. □

Remark: Only the existence of solutions of both Type I and Type II in the fast diffusion case appears in [27].

Comments on the variation of $\bar{\beta}_0$. We are not able to establish the number of minima and maxima of the anomalous curve $\bar{\beta}_0$ in the range $p\in(1,p_s)$. The numerical experiments presented below (see Figure 2.3) suggest that there is a unique minimum point. In order to look for these points, one has to differentiate with respect to m in the second equation of (2.86) and obtain:

$$-m(m+1)\beta_0'(m)=\beta_0(m), \tag{2.87}$$

at all the points $m=p-1\in(0,m_s)$ that are critical for $\bar{\beta}_0$.

Conjecture: The function $\beta_0(m)$ of the PME case is a convex function of m.

If the conjecture were true, then $\beta_0'(m)$ would be a positive and increasing function of m, hence the left-hand side of (2.87) would be decreasing. But it is well-known (see [127]) that β_0 is an increasing function of m. Hence, the equation (2.87) will have a unique solution $m^* \in (0, m_c)$, and $p^* = 1 + m^* \in (1, p_c)$ will be the unique minimum point of $\bar{\beta}_0$, as the numerical experiments indicate (see Figure 2.3 below).

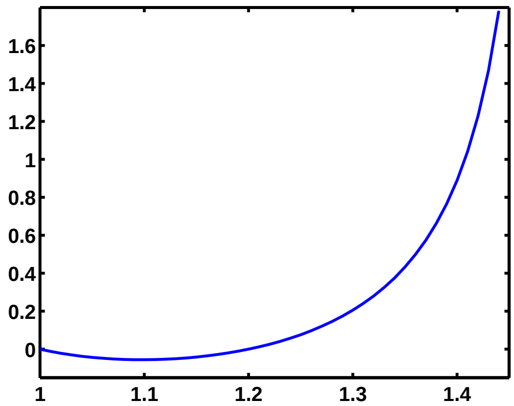

Figure 2.3: The dependence of $\bar{\beta}$ on p, for $n = 3$.

2.9 Main results for $m < 0$ and $p < 1$

In the following sections we extend to the very fast diffusion case the correspondences relating the radial solutions of the two basic models of nonlinear diffusion that we are considering. More precisely we deal with the complementary cases $m < 0$, respectively $p < 1$. We will also consider the limit case $m = -1$, $p = 0$, which has particular importance in practical applications and is also interesting from the mathematical point of view.

Why very fast diffusion? Completing the models described or quoted in the Subsection 1.1.1, we present some models from applied sciences, more precisely from physics, chemistry and treatment of images, which justify the mathematical study of the equations that we generally refer as very fast diffusion equations. We also make references to the mathematics of the fast diffusion.

The standard very fast diffusion equation, (VFDE). Under this name, we understand the very fast diffusion version of the PME, that is:

$$u_t = \mathrm{div}(u^{m-1}\nabla u), \quad m < 0, \tag{2.88}$$

posed in $\mathbb{R}^n$ (in all the following sections of this chapter we will be interested only in equations posed on the whole space). This equation was proposed by G. Rosen as a model for the nonlinear heat conduction in solid hydrogen atoms, see [117], where the equation for $m = -1$ is deduced experimentally. After that, Chayes, Osher and Ralston have proposed this model for $m < 0$ and $n = 1$ to model avalanches in sandpiles, see [41]. The VFDE in general is also used by Meerson (see [104]) to describe the cooling of a fireball produced by a strong explosion of a local gas. Some special cases in relation with it are studied also by Kamin and Dascal in [87].

From the mathematical point of view, the VFDE started to be studied in detail due to its unusual properties with respect to other types of parabolic equations, first by Bertsch, Luckhaus, Dal Passo and Ughi in several papers, see for example [48], [25] and [26]. For general qualitative results about (2.88) we refer to Chapter 9 of the recent book [127] and to previous research papers, as [74], [114], [121], [125].

The very fast p-Laplacian equation (FPLE). Under this name we understand the singular equation

$$u_t = \frac{1}{p-1}\Delta_p u, \quad p < 1, \tag{2.89}$$

where we divide by $p-1$ in order to have a monotone increasing function $\Phi(s) = (p-1)^{-1}|s|^{p-2}s$ for $p < 1$, hence a parabolic equation in radial variables, cf. convention in [115] or [14].

This equation was proposed by Barenblatt and Vázquez in [14] as a particular case in a model for image contour enhancement, which is a phenomena of great interest in applications of denoising and recognition of images, see also [13]. From this model the FPLE with $p < 0$ arises, in dimension $n = 1$, and some other more general variants of it. Some parallel models appear in [2].

There are very few mathematical papers dealing with this equation. Nevertheless, some strange and unexpected phenomena were revealed by Rodriguez and Vázquez in [115], where it is studied the FPLE only in dimension $n = 1$. In the chapter we study, through the correspondence relations we establish, the radial and, in particular, self-similar solutions of FPLE in all dimensions, and we give also various explicit examples of solutions.

Note. There exist many other models in image processing using the different (although similar in writing) equation

$$u_t = \Delta_p u, \ p < 1, \tag{2.90}$$

i. e. without dividing by $p-1$. For example, Keeling and Stolberger (see [91]) use (2.90) with $p = 0$ to construct a filter in order to denoise images and preserve details, filter called *the balanced forward-backward filter*. After that, (2.90) was used in many experiments (see [133] and [105]). The equation (2.90) is very different with respect to the FPLE, since it is not parabolic in radial variables and does not diffuse in this direction, but it is parabolic in angular variables, while the FPLE has opposite properties. Nevertheless, for self-similar solutions there is an immediate transformation from the FPLE to (2.90).

Some remarks. (i) We make the convention, also introduced in [78] and [79], that the dimensions and variables concerning the FPLE are named similarly as the correspondent ones for the VFDE, but with an overline. Since we deal only with radial solutions and variables, we will also accept noninteger dimensions n and $\bar{n}$. We perform our general analysis of the VFDE (in Section 2.12) using n as a **real** parameter. This setting will be very useful when passing to the analysis of the FPLE through the Theorems above, since integer values of $\bar{n}$ might come from noninteger values of n.

(ii) In [78], the values of the constants D_1 and D_2 in Theorems 2.1 and 2.2 are different from those of [79], since in the former we start from the PLE written in the usual form, without dividing by $p-1$, while in the latter we divide by $p-1$. Keeping this convention in mind, in the rest of the chapter we will consider the values of the constants considered in [79], namely

$$D_1 = \left(\frac{(2m)^2}{(m+1)^2}\right)^{\frac{1}{m-1}}, \quad D_2 = \left(\frac{(mn-n+2)^2}{(m+1)^2}\right)^{\frac{1}{m-1}}, \tag{2.91}$$

which lead to slight changes in the formulas with respect to the previous sections, which were already considered in the paper [79].

(iii) As important examples of radial solutions for the equations we deal with, there are the radially symmetric self-similar solutions, that can take one of the three forms indicated in (2.11), called respectively self-similarity of type I, II or III.

(iv) **Self-similarity for** (2.90). At the level of self-similar solutions, there exists a direct transformation from (2.89) into (2.90). Indeed, in order to pass from self-similar solutions of the FPLE to self-similar solutions of (2.90), it suffices to interchange Type I with Type II and to change at the same time the exponent β into $-\beta$.

2.9.1 Outline of results

1. Correspondence of radial solutions of VFDE and FPLE. Starting from Theorems 2.1 and 2.2, we show that these relations still hold for $m < 0$ and $p < 1$, but only in some conditions, that we formulate in the next statement:

Theorem 2.8. *(a) For $-1 < m < 0$ and $0 < p < 1$, the radially symmetric solutions of VFDE and FPLE are related through the transformations in Theorem 2.1.*

(b) For $m < -1$ and $p < 0$, to any dimension $\bar{n}$ associated to the FPLE, there correspond two dimensions n_1 and n_2 associated to the VFDE, with $2 < n_1 < \infty$ and $2/(1-m) < n_2 < 2$ respectively. The two transformations (2.2) *of Theorem 2.1 and* (2.4) *of Theorem 2.2 hold. In particular, we also derive a complete self-map of the VFDE.*

(c) For $m = -1$ and $p = 0$, there exists a correspondence between the radially symmetric solutions of VFDE in dimension $n = 1$ and radially symmetric solutions of the FPLE in any dimension $\bar{n} > 0$, given by

$$\bar{u}_{\bar{r}} = \frac{1}{\bar{n}} r^{1-\frac{1}{\bar{n}}} u, \quad \bar{r} = r^{1/\bar{n}}. \tag{2.92}$$

Consequently, we obtain a complete self-map for the FPLE with $p = 0$.

We prove this theorem in Sections 2.10 and 2.11.

2. Applications: self-similar solutions for the two equations. We want to study the self-similar solutions of the FPLE. In order to do this, we perform first a complete and detailed study of the self-similar solutions of VFDE, extending in this way in all dimensions $n > 0$ the results obtained for $n = 1$ by Ferreira and Vázquez in [60]. We prove that the self-similar solutions of the VFDE can have only the following types of behavior as $\eta \to \infty$ or as $\eta \to 0$:

$$f(\eta) \sim C\eta^{\frac{2-n}{m}}, \; f(\eta) \sim C\eta^{-\frac{\alpha}{\beta}}, \; f(\eta) \sim C\eta^{-\frac{2}{1-m}}, \; f(0) = C \text{ with } f'(0) = 0,$$

Moreover, there exist blow-up or large solution profiles (i. e. such that $f(\eta) \to \infty$ as $\eta \to \eta_0 \in (0,\infty)$, with $\eta < \eta_0$, resp. $\eta > \eta_0$) and in critical cases as $m = m_c := (n-2)/n$ or $\beta = 0$, we find profiles presenting logarithmic corrections in their behavior, more precisely $f(\eta) \sim C(\eta\sqrt{|\log\eta|})^{-2/(1-m)}$. These results are made precise in Subsection 2.12.3 as Theorems 2.10, 2.11 and 2.12.

Then, we translate the results to the FPLE through Theorem 2.8. We obtain in this way a similar list of rates of behavior for $p \neq 0$:

$$\bar{f}(\bar\eta) \sim C\bar\eta^{\frac{p}{p-1}} + K(C), \; \bar{f}(\bar\eta) \sim C\bar\eta^{-\frac{\bar\alpha}{\bar\beta}}, \; \bar{f}(\bar\eta) \sim C\bar\eta^{-\frac{p}{2-p}}, \; \bar{f}(\bar\eta) \sim C\bar\eta^{\frac{p-\bar n}{p-1}} + C_2,$$

and in some critical cases again profiles with logarithmic corrections, precisely: $\bar{f}(\bar\eta) \sim C(\bar\eta^p \log\bar\eta)^{1/(p-2)}$. Finally, we identify profiles that present the phenomenon of *quenching* (i. e. blow-up of the gradient of the solution), without blowing-up at some intermediate point, which are also qualitatively interesting. We also obtain many explicit solutions for both equations, see Subsection 2.12.4 for the VFDE and Section 2.13 for the FPLE. As already explained, this implies also a classification of the self-similar solutions of (2.90).

We treat separately the limit case $m = -1$, $p = 0$, which is different from the rest, using part (c) of Theorem 2.8. The analysis of this case is formalized as Theorem 2.13 and done in Section 2.13.

3. Qualitative consequences. The study of self-similarity for both equations allows us to derive some qualitative results about them.

(a) It is easy to see from the results of Section 2.12 that the self-similar solutions of the VFDE can not have change of sign. In fact, a more careful analysis shows that the changing sign solutions, which exist in the standard case $m > 1$, are replaced by blow-up solutions.

(b) As a consequence, from Theorem 2.8 we deduce that the self-similar profiles of the FPLE are necessarily monotonic. There is also a special type of solutions of the FPLE which present quenching, but without blowing-up.

(c) We perform in Section 2.14 a study of **self-similar solutions of the FPLE with finite mass** (i. e. $\|f\|_1 < \infty$) which proves that such solutions exist in very general situations, contrary of what happens in the VFDE case. We gather the results as

Theorem 2.9. *(i) For $p < 0$, there exist self-similar solutions of type I with finite mass having $\bar\beta \in (0, -1/(\bar n(p-2)+p))$. For $0 < p < 1$, there exist also solutions of type I with finite mass for $p \in (\bar n/(1+\bar n), 1)$. In particular, in this latter case, for any such p and $\bar n$ there exists a solution providing* mass conservation, *having $\bar\beta = 1/(\bar n(p-2)+p)$ and the behavior rates $\bar f(\bar\eta) \sim \bar C_0\bar\eta^{-p/(2-p)}$ as $\bar\eta \to 0$ and $\bar f(\bar\eta) \sim C\bar\eta^{p/(p-1)}$ as $\bar\eta \to \infty$.*

(ii) There are no finite mass self-similar solutions of type II for $p < 0$. For $p > 0$, there exist self-similar solutions of type II with finite mass for $p \in (\bar n/(1+\bar n), 1)$. In particular, for any such p and $\bar n$, there exists a solution of type II with mass conservation before extinction time,

having $\bar{\beta} = -1/(\bar{n}(p-2)+p)$, starting with $\bar{f}(0) = C$ and behaving like $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{p/(p-1)}$ as $\bar{\eta} \to \infty$.
(iii) For $p = 0$, there are no type II self-similar solutions with finite mass, and there exist solutions of type I with finite mass for $\bar{\beta} \in (0, 1/2\bar{n})$.

4. A more specific application: level connecting profiles of the FPLE. We study in Section 2.15 the profiles that we call level connecting, i. e. those profiles which connect two constant levels, for example those having $\bar{f}(0) = 0$ and $\lim_{\bar{\eta}\to\infty} \bar{f}(\bar{\eta}) = C$ or the profiles having $\bar{f}(0) = C$ and having quenching at some finite point. This type of profiles is relevant in applications in image processing, as showed for example in [14] and explained in more detail at the beginning of the Section 2.15.

2.10 Radial correspondence relations and self-maps for $p \neq 0$

Our aim is to prove parts (a) and (b) of Theorem 2.8. We start from Theorem 2.2 and we study which of the transformations (2.2) and (2.4) applies, depending on m and p. We differentiate two cases with respect to the sign of $\bar{n}_1$ and $\bar{n}_2$. We avoid for the moment the special case $m = -1$, $p = 0$, which requires a different treatment, that we develop in the next section.
Case 1: $-1 < m < 0$ **and** $0 < p < 1$. We observe that $\bar{n}_1 < 0$, while $\bar{n}_2 > 0$ if and only if $n < 2$ and $m > m_c := (n-2)/n$. In this case, we remark that $\bar{n}_2$ ranges from 0 to ∞ while $n \in (0, 2/(1-m))$. The correspondence has a vertical asymptote at $n = 2/(1-m) \in (1, 2)$, then $\bar{n}$ becomes negative and it remains always below 1, hence this other curve does not bring any novelty. We draw the correspondence of the dimensions in Figure 2.4 below, where the numerical experiment is performed with $m = -1/2$ and the vertical asymptote appears at $n = 4/3$.

Case 2: $m < -1$ **and** $p < 0$. We observe that $\bar{n}_1 > 0$, while $\bar{n}_2 > 0$ if and only if $m < m_c$, equivalently $n \in (n_0, 2)$, where $n_0 = 2/(1-m)$. Since $m < -1$, we remark that $n_0 \in (0, 1)$. It is easy to see that $\bar{n}_2$ ranges between 0 and ∞ in a linear way, for $n \in (2/(1-m), 2)$. In conclusion, we cover all the dimensions $\bar{n} \in (0, \infty)$ in the following two ways:
(i) By $n \in (2, \infty)$, and there holds the correspondence relations given in part (i) of Theorem 2.2;
(ii) By $n \in (2/(1-m), 2)$, and there hold the correspondence relations given in part (ii) of Theorem 2.2. We draw the correspondence of the dimensions in this case in Figure 2.5 below, where the numerical experiment is performed for $m = -3/2$ and the vertical asymptote appears at $n = 4/5$.

Due to these correspondences, we obtain for $m < -1$ a **complete self-map** of the VFDE, in contrast to the standard case $m > 0$, where the self-map holds true only in some particular ranges (see Section 2.3). In our case, any dimension $\bar{n} \in (0, \infty)$ is obtained from some $n_1 \in (2, \infty)$ and from $n_2 \in (2/(1-m), 2)$. By equating $\bar{n}_1$ and $\bar{n}_2$, we find that the self-map

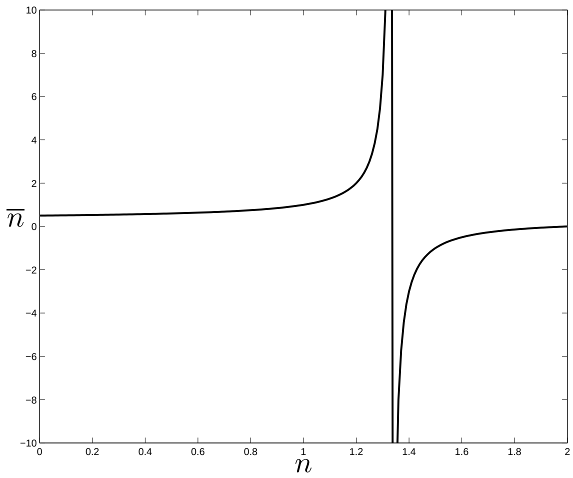

Figure 2.4: Correspondence of dimensions for $m = -1/2$, $p = 1/2$.

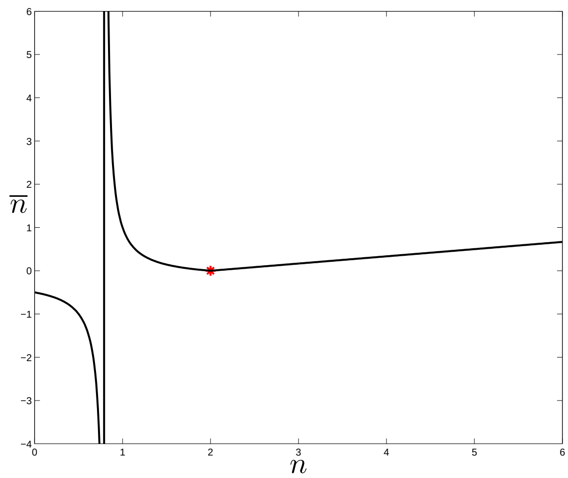

Figure 2.5: Correspondence of dimensions for $m = -3/2$, $p = -1/2$

is given by

$$u_1(r_1,t) = \frac{D_2}{D_1} r_2^{\frac{n_2-2}{m}} u_2(r_2,t), \quad r_1 = r_2^{\frac{mn_2-n_2+2}{2m}}, \quad n_1 = \frac{2(n_2-2m-2)}{n_2-mn_2-2},. \tag{2.93}$$

The same self-map holds also for $-1 \le m < 0$, but with the difference that here the self-map does not pass through the correspondence with the FPLE, but it appears through the part where this correspondence does not hold. The unique fixed point of this self-map is $n_1 = n_2 = 2$. This self-map was first established by King in [95] for the more "standard" range $m > 0$.

2.11 The special case $m = -1$ and $p = 0$

We prove part (c) of Theorem 2.8. Let us notice that in this limit case, the transformations in Theorem 2.2 do not make sense in the same form. But we observe that the vertical asymptote of the two cases above approaches $n = 1$ as $m \to -1$ (see Figures 2.4 and 2.5). This suggests us to look for a correspondence relation with $n = 1$ and with $\bar{n}$ any real number. We thus consider the following:

$$\bar{u}_{\bar{r}}(\bar{r}, t) = \bar{n} r^{1-1/\bar{n}} u(r, t) \qquad \text{where} \qquad \bar{r} = r^{1/\bar{n}}. \tag{2.94}$$

We start with $\bar{u}$ as a solution of FPLE and calculate:

$$\begin{aligned} \bar{u}_{\bar{r},t} &= -\frac{\partial}{\partial \bar{r}}\left(\bar{r}^{1-\bar{n}} \frac{\partial}{\partial \bar{r}}\left(\bar{r}^{\bar{n}-1}|\bar{u}_{\bar{r}}|^{-2}\bar{u}_{\bar{r}}\right)\right) = -\frac{1}{\bar{n}}\frac{\partial}{\partial \bar{r}}\left(r^{(1-\bar{n})/\bar{n}}\frac{\partial |u|^{-2}u}{\partial \bar{r}}\right) \\ &= \bar{n} r^{1-1/\bar{n}} \frac{\partial}{\partial r}\left(|u|^{-2}u_r\right). \end{aligned}$$

On the other hand, directly from (2.94), we deduce that

$$\bar{u}_{\bar{r},t} = \bar{n} r^{1-1/\bar{n}} u_t.$$

Finally, equalizing the two calculations and simplifying, we deduce that u is a solution of the VFDE with $m = -1$ in dimension $n = 1$. Of course, conversely, the correspondence is unique up to summing scalar constants, since the invariance with respect to addition of constants is an obvious feature of the PLE.

As an immediate consequence, if we consider $\bar{n}_1, \bar{n}_2 > 0$, we can correspond radial solutions of the FPLE with $p = 0$ in dimensions $\bar{n}_1$ and $\bar{n}_2$. Indeed, if we start with the same solutions u of the VFDE, we easily find that

$$\bar{r}_1^{1-\bar{n}_1}\bar{u}_{1,\bar{r}_1} = \bar{r}_2^{1-\bar{n}_2}\bar{u}_{2,\bar{r}_2}, \quad \bar{r}_1^{\bar{n}_1} = \bar{r}_2^{\bar{n}_2}$$

or, equivalently, $\bar{u}_{2,\bar{r}_2} = \bar{r}_1^{(\bar{n}_2-\bar{n}_1)/\bar{n}_2}\bar{u}_{1,\bar{r}_1}$.

Thus, we obtain a **complete self-map** for the PLE with $p = 0$; this is a unique property, since we have remarked that for other values of p we have no such self-maps of the PLE. This ends the proof of Theorem 2.8.

The Backlund transform. It is well-known that, for dimension $n = 1$, there exists a transformation from the VFDE with $m = -1$ into the heat equation. This is called the Backlund transform and appears for example in [117]. Start with a solution $u > 0$ of the VFDE with $m = -1$, in dimension $n = 1$, and consider the function:

$$X(x, t) = \int_{x_0}^{x} u(x, t)\,dx + \int_{t_0}^{t} (u^{-2}u_x)|_{x=x_0}\,dt$$

for (x_0, t_0) fixed. It is easy to check that the transformation $T:(x,t)\mapsto(X,t)$ is smooth and invertible and has the Jacobian exactly u. Take $x=w(X,t)$ the inverse, then w is a solution of the heat equation. In this way, we can relate the FPLE with $p=0$ in any dimension $\bar{n}$ with the classical heat equation. For more details and applications of the Backlund transform, see [126].

Remark: in [126], it is shown that X introduced above is itself a solution of the FPLE with $p=0$, in dimension $\bar{n}=1$. But our transforms are more general, since we correspond the VFDE with $m=-1$ in dimension $n=1$ with the FPLE with $p=0$ in any dimension $\bar{n}$.

2.12 Self-similar solutions for the very fast diffusion equation

We perform a complete and detailed study of self-similarity for the VFDE. Consider self-similar solutions of the three types indicated in (2.11) and recall that, by associating the following variables:

$$\Phi=(2+(1-m)\eta f^{'}/f)/\sqrt{|b|},\quad \Psi=\eta^2|f|^{1-m}/|b|,$$

where $b=2n(m-m_c)/(m-1)$, provided that $b\neq 0$, we arrive to the two-dimensional quadratic autonomous dynamical system:

$$\left\{\begin{array}{l}\dot{\Psi}=\Psi\Phi,\\ \dot{\Phi}=c_1\Phi^2-c_2\Psi\Phi-c_3\Phi+\varepsilon\Psi+\mathrm{sgn}(b),\end{array}\right. \tag{2.95}$$

where over-dot indicates differentiation with respect to $\sqrt{|b|}\log\eta$. The value of ε indicates the type of self-similarity through the relation between the coefficients: we have $(m-1)\alpha+2\beta=\varepsilon$, where the solution is of type I, II or III if $\varepsilon=1$, $\varepsilon=-1$ and $\varepsilon=0$ respectively. The detailed deduction is given in Section 2.4. We recall only the values of the coefficients c_i with $i=1,2,3$:

$$c_1=m/(m-1),\quad c_2=\beta\sqrt{|b|},\quad c_3=\frac{\sqrt{2}}{2}\frac{mn+2m-n+2}{(m-1)\sqrt{|(mn-n+2)/(m-1)|}}. \tag{2.96}$$

From the expression of c_3, we remark that we will have a bifurcation in the behavior of the phase-plane system at $m=-1$.

Note. From the results in Section 2.10, it is enough to perform the study of the phase-plane in dimensions $n<2$, since it may be then transported to any dimension through the complete self-map obtained there. Also, the analysis in $n<2$ is sufficient to obtain the whole information on the FPLE. Due to these remarks, we will concentrate on the study of the case $n<2$ for all values of $m<0$. We will postpone the analysis on the special case $m=m_c$, i. e. $b=0$, where we use a different system.

2.12.1 Analysis of the critical points

Analysis of the critical points in the plane. For $\Psi=0$ we have two critical points: $P_1=(0,\mathrm{sgn}(b)\sqrt{|b|}/2c_1)$ and $P_2=(0,2/\sqrt{|b|})$, lying on the y-axis. For $\Phi=0$, since we are interested only in points with $\Psi > 0$ (in the upper half-plane) we find a critical point $P_3 = (1,0)$ if and only if $\mathrm{sgn}(b)\varepsilon = -1$. Let us recall here that in our case $c_1 = m/(m-1) \in (0,1)$. Analyzing these (possible) three points, we obtain that:

- P_1: the linearized system around P_1 has eigenvalues $\lambda_1 = \mathrm{sgn}(b)\sqrt{|b|}/2c_1$ and $\lambda_2 = (n-2)/\sqrt{|b|}$. Since $c_1 > 0$, if $\mathrm{sgn}(b) = 1$, this point is a saddle, while in the contrary case, if $\mathrm{sgn}(b) = -1$, this point is an attractor.
- P_2: the linearized system around P_2 has eigenvalues $\lambda_1 = 2/\sqrt{|b|}$ and $\lambda_2 = -(n-2)/\sqrt{|b|}$. Since we are treating here only the case $n < 2$, this is always a repeller.
- P_3: the linearized system around P_3 has eigenvalues

$$\lambda_{1,2} = (-(c_2+c_3) \pm \sqrt{(c_2+c_3)^2 + 4\varepsilon})/2.$$

The type of this point depends on ε and $c_2 + c_3$ in the following way: if $\varepsilon = 1$ it is a saddle point, but for $\varepsilon = -1$ (which characterizes the self-similarity of Type II, the most interesting in our case), this point could be: a stable node if $c_2+c_3 \geq 2$, an unstable node if $c_2+c_3 \leq -2$, a stable spiral if $0 < c_2 + c_3 < 2$, an unstable spiral if $-2 < c_2 + c_3 < 0$ or even a center if $c_2 + c_3 = 0$. In the case that this point is a center in the linearized system, in the original system this could be a center or a spiral. In order to classify this, we use a general result appeared in Section 2 of [46]. We translate P_3 at the origin by setting $H = \Psi - 1$ and, since $c_3 = -c_2$, the equation of the trajectories of the system (2.95) is

$$\frac{d\Phi}{dH} = -\frac{H + c_2 H\Phi - c_1\Phi^2}{\Phi + H\Phi}.$$

Using results from [46], we find that P_3 is a center in the nonlinear system if and only if $c_2 = c_3 = 0$. In the other cases, even if $c_2 + c_3 = 0$, but $c_2 \neq 0$, this point is a spiral for (2.95). In any case, when this point lies in the half-plane $\Psi > 0$, its analysis will be more involved.

Analysis of the critical points at infinity. In order to perform this analysis, we use the standard procedure of transforming our plane into the Poincaré Sphere by introducing the homogeneous coordinates (see for example [109])

$$\Psi = \frac{U}{W}, \quad \Phi = \frac{V}{W}, \quad U^2 + V^2 + W^2 = 1,$$

and the system (2.95), written in differential form, becomes:

$$\begin{aligned} &WUVdV + (-c_1V^2W + c_2UVW + c_3VW^2 - \varepsilon UW^2 - \mathrm{sgn}(b)W^3)dU \\ &\quad + ((c_1-1)V^2U - c_2U^2V - c_3UVW + \varepsilon U^2W + \mathrm{sgn}(b)W^2U)dW = 0. \end{aligned} \tag{2.97}$$

The critical points at infinity correspond to points on the equator of the sphere satisfying the following conditions:

$$U \geq 0, \quad W = 0, \quad U^2 + V^2 = 1, \quad UV((c_1-1)V - c_2U) = 0,$$

hence we obtain four such points:

$$Q_1 = (1,0,0), \quad Q_2 = (\frac{1-c_1}{\sqrt{(c_1-1)^2 + c_2^2}}, -\frac{c_2}{\sqrt{(c_1-1)^2 + c_2^2}}, 0), \quad Q_{3,4} = (0, \pm 1, 0).$$

In order to analyze separately these points, we consider the projection of (2.97) on the plane $U = 1$, obtaining the system:

$$\begin{cases} \dot{W} = -VW, \\ \dot{V} = -(1-c_1)V^2 - c_2V - c_3WV + \varepsilon W + \mathrm{sgn}(b)W^2, \end{cases} \tag{2.98}$$

where the first two critical points transform into $Q_1 = (0,0)$, $Q_2 = (0, c_2/(c_1-1))$. We are now in position to proceed with the separate analysis of the points.

• Q_1: the linearized system around Q_1 has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -c_2$, hence this is a degenerate critical point. Using the methods of Section 2.12 in the book [109], we remark that Q_1 is a saddle-node whose center manifold has the form $V = \varepsilon W/c_2 + o(W)$ and the flow on this manifold is given by $W' = -\varepsilon W^2/c_2 + o(W^2)$. We conclude that in the region $W > 0$, the point behaves like a node for $\varepsilon = 1$, which is stable if $c_2 > 0$ and unstable if $c_2 < 0$, and as a saddle if $\varepsilon = -1$. Moreover, in the latter there exists a unique orbit in the half-plane $W > 0$, which enters Q_1 if $c_2 < 0$ and comes from Q_1 if $c_2 > 0$. The analysis for $\varepsilon = 0$ will be performed at its place.

• Q_2: the linearized system around Q_2 has eigenvalues $\lambda_1 = -c_2/(c_1-1)$ and $\lambda_2 = c_2$. Since in our case $c_1 \in (0,1)$, Q_2 is a stable node if $c_2 < 0$ and an unstable node if $c_2 > 0$. If $c_2 = 0$, then $Q_2 = Q_1$.

• Q_3 **and** Q_4: in order to analyze these points, we consider the projection of (2.97) on the planes $V = \pm 1$ respectively, which transforms our points into the origin of the projected plane, and the equation (2.97) becomes the following system:

$$\begin{cases} \pm\dot{W} = c_1W \mp c_2UW \mp c_3W^2 + \varepsilon UW^2 + \mathrm{sgn}(b)W^3, \\ \pm\dot{U} = -(1-c_1)U \mp c_2U^2 \mp c_3UW + \varepsilon U^2W + \mathrm{sgn}(b)W^2U, \end{cases} \tag{2.99}$$

in view of Theorems 1 and 2, Section 3.10 of [109]. This system, linearized around the origin, has eigenvalues $\lambda_1 = c_1$ and $\lambda_2 = c_1 - 1$, resp. $\lambda_1 = -c_1$ and $\lambda_2 = 1 - c_1$. Hence Q_3 and Q_4 are always saddle points. Since $U \equiv 0$ and $W \equiv 0$ are trajectories of the system (2.99), there are no other trajectories passing through these points.

Existence and uniqueness of limit cycles. We have:

Proposition 2.3. *The system (2.95) has a limit cycle if and only if $\varepsilon = -1$, sgn$(b) = 1$ and*

$$\beta_1^* := -\frac{(n+2)(m-m_s)}{2(mn-n+2)} < \beta < \beta_2^* := -\frac{m}{mn-n+2},$$

and in this case the limit cycle is unique.

Proof. Denote $P = \Psi\Phi$ and $Q = c_1\Phi^2 - c_2\Psi\Phi - c_3\Phi + \varepsilon\Psi + \mathrm{sgn}(b)$, which are quadratic polynomials in Ψ and Φ. We divide the proof into several steps:

(a) It is clear that a limit cycle of (2.95) should contain the point P_3 in the interior. Moreover, it follows from results in [46] that the limit cycle only can contain a focus in its interior. On the other hand, Theorem 6.4, page 275 in the book [136] implies that (2.95) has at most one

limit cycle. From this, we deduce that the limit cycle may exist only if $\varepsilon = -1$, $\text{sgn}(b) = 1$ and $|c_2 + c_3| < 2$.
(b) The vectors of the direction field of (2.95) over the line $r : \Phi = \sqrt{b}/2c_1$ have the same direction. Let $V = (0, 1)$ be the normal vector of r. We calculate:

$$(P, Q) \cdot V = -\Psi(-c_2\sqrt{b}/2c_1 + 1).$$

It follows that if $c_2 > -2c_1/\sqrt{b}$, all these vectors have contrary direction to V, if $c_2 < -2c_1/\sqrt{b}$ all these vectors have the same direction of V and if $c_2 = -2c_1/\sqrt{b}$ then r is an explicit trajectory.
(c) We set $H = \Psi - 1$ in order to translate the point P_3 at the origin. We obtain:

$$\left\{ \begin{array}{l} \dot{H} = \Phi + H\Phi = R_1, \\ \dot{\Phi} = -H + \mu\Phi + c_1\Phi^2 + (\mu + c_3)H\Phi = S_1, \end{array} \right. . \tag{2.100}$$

where $\mu = -(c_2 + c_3)$. The Lyapunov number of (2.100) is $\sigma = 3\pi c_3/2$ (see [109], page 344). Using the Hopf Bifurcation Theorem, we obtain that there exists a unique unstable limit cycle bifurcating from the origin as μ increases from 0.
(d) The system (2.100) is a *semicomplete family (mod $R_1 = 0$) of rotate vector field (SRF)* with parameter μ, i. e. R_1 does not depend on μ and it satisfies the following on the set where $R_1 \neq 0$:
• The critical points are fixed as $\mu \in \mathbb{R}$;
• $R_1(\partial S_1/\partial\mu)/(R_1^2 + S_1^2) > 0$ for $\mu \in \mathbb{R}$;
• If $\theta = \arctan(S_1/R_1)$, then $\tan\theta \to \pm\infty$ as $\mu \to \pm\infty$.
Using results in [110] or in the appendix of [45], we obtain:
(i) Limit cycles of distinct fields of (2.100) do not intersect.
(ii) Stable and unstable limit cycles of (2.100) expand or contract monotonically as μ varies in a fixed sense. The motion covers an annular neighborhood of the initial position.
(iii) A semistable limit cycle of (2.100) splits into a stable limit cycle and an unstable limit cycle if μ is varied in a suitable sense, while it disappears if μ is varied in the opposite sense.
(iv) Let $L(\mu)$ be a limit cycle of (2.100) and R the region covered by it as μ varies in $\mathbb{R}$. Then the inner (outer) boundary of R consists of either a single point, a separatrix cycle or a semistable limit cycle.
(e) **Conclusions**. From the previous discussion, we deduce that there exists a unique limit cycle, which generates when $c_2+c_3 = 0$, then increases until being absorbed by the separatrix cycle formed by P_1, Q_1, Q_4, the line connecting P_1 and Q_1 (denoted by r), the line connecting P_1 and Q_4 ($\Psi = 0$) and the arc connecting Q_1 and Q_4, contained in the projection of the Poincaré sphere on the half-disk $\{U \geq 0,\ W = 0,\ U^2 + V^2 \leq 1\}$. This holds when $c_2 + c_3 = \sqrt{b}/2$; moreover, from steps (b) and (d), there is no limit cycle for $c_2 + c_3 < 0$, neither for $c_2 + c_3 > \sqrt{b}/2$, and we are done. □

2.12.2 Analysis in terms of profiles

We transform in terms of profiles the local analysis performed before. As we have seen from the analysis of the critical points, there are three parameters that vary their behavior, which are $\varepsilon \in \{-1,0,1\}$, giving the type of self-similarity, then $\mathrm{sgn}(b)$ and the coefficient c_2. We will analyze all these cases, assuming for the moment that $c_2 \neq 0$. The special case $c_2 = 0$ will be treated separately.

Case 1: $\varepsilon = 1$ **and** $\mathrm{sgn}(b) = 1$. In this case, the point P_3 does not appear.

• For the point P_1, we obtain that $\Phi \to \frac{\sqrt{b}(m-1)}{2m}$, and performing direct calculations we find that the orbits going through P_1 are characterized by $f(\eta) \sim C\eta^{(2-n)/m}$ as $\eta \to 0$. Hence these are solutions developing a singularity as $\eta \to 0$.

• The analysis of P_2 is more involved. The linearized system near P_2 has eigenvalues $\lambda_1 = 2/\sqrt{b}$ and $\lambda_2 = -(n-2)/\sqrt{b}$, with corresponding eigenvectors $e_1 = (n, -2c_2 + \sqrt{b})$ and $e_2 = (0,1)$. There is one orbit tangent to e_1, this is given by profiles with the property $f(\eta)^{m-1} \sim (2\beta-1)\eta^2/2n + C_2$ as $\eta \to 0$, hence $f(\eta) \sim C_2$ and $f^{'}(\eta) \sim 0$. All the other orbits are tangent to e_2, i. e. to the x-axis. These orbits contain profiles behaving like $f(0) = C > 0$. Looking for their second approximation, we find that they satisfy $\Phi = 2/\sqrt{b} + k\Psi^a$, with either $a = 1$ (which gives us the orbit above), or $a = (2-n)/2$. The orbits tangent to the x-axis, with $a = (2-n)/2$, can be written as $f(\eta) = C_1 + C_2\eta^{2-n} + o(\eta^{2-n})$ as $\eta \to 0$. We remark that the local analysis near P_2 is similar in all cases with $b \neq 0$, hence we will omit it in the rest.

• For the point Q_1, we start with $\Phi \sim 1/c_2$ and we find that, for $c_2 > 0$, the orbits that enter this point contain the profiles with $f(\eta) \sim \eta^{-\alpha/\beta}$ as $\eta \to \infty$. For $c_2 < 0$ the analysis reverses and the orbits that come from Q_1 contain the profiles with $f(\eta) \sim \eta^{-\alpha/\beta}$ as $\eta \to 0$.

• For the point Q_2, we have $V/U = \Phi/\Psi \sim c_2/(c_1-1)$. By transforming this and integrating, we find that the orbits entering (for $c_2 < 0$) or coming from this point (for $c_2 > 0$) consist in profiles f behaving like $f(\eta) \sim \left(C - k\eta^2\right)^{1/(m-1)}$ with $k > 0$ and arbitrary $C > 0$, i. e. having a blow-up point $r_0 \in (0,\infty)$ in both cases. We also remark that the local analysis of this point is independent on ε and b, hence it will be the same in all cases. Hence, there is only one point where the trajectories enter, which is Q_1 if $\beta > 0$ and Q_2 if $\beta < 0$. It follows that the phase-plane is topologically equivalent in this case to the one indicated in Figure 2.6 below.

Case 2: $\varepsilon = 1$ **and** $\mathrm{sgn}(b) = -1$. In this case, the point P_3 appears. We also remark that the analysis near P_2, Q_1 and Q_2 is identical to the previous case.

• For the point P_1, we obtain that $\Phi \to -\sqrt{b}/2c_1$ as $\eta \to \infty$. The orbits through P_1 contain the profiles with $f(\eta) \sim C\eta^{(2-n)/m}$ as $\eta \to \infty$.

• The point P_3 is a saddle point. Starting from $\Psi \to 1$, we find that the orbits coming or entering this point contain the profiles satisfying $f(\eta) \sim C\eta^{-2/(1-m)}$ as $\eta \to 0$ for the orbits coming out of P_3 or as $\eta \to \infty$ for the orbits entering P_3. On the other hand, since P_3 is itself a solution (see Subsection 2.12.4, part (b)), we find that the constant C has a precise

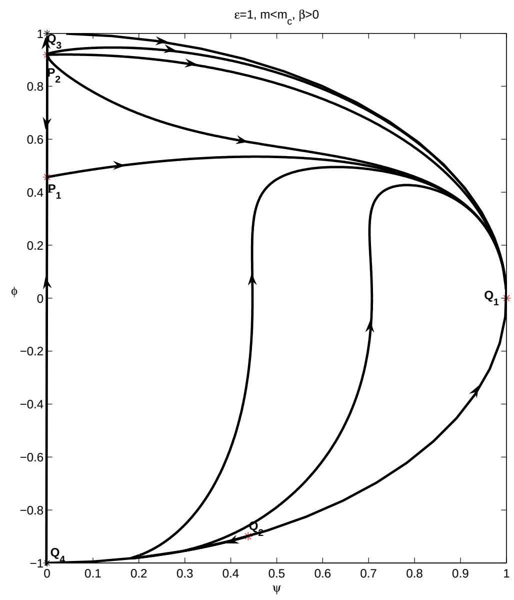

Figure 2.6: Phase portrait for Case 1. Experiment for $m = -5$, $n = 0.7$, $\beta = 0.4$.

value:

$$C_0 = \left(-\frac{(m-1)\varepsilon}{2(mn-n+2)} \right)^{\frac{1}{m-1}}.$$

The four separatrices of the saddle point P_3 connect with each one of the points P_1, P_2, Q_1 and Q_2, hence the phase-plane is topologically equivalent to the one indicated in Figure 2.7.

Case 3: $\varepsilon = -1$ **and** $\mathrm{sgn}(b) = 1$. The point P_3 appears. The local analysis near the points P_1, P_2 and Q_2 is identical to Case 1.

• The point P_3 has a variable type depending on the sum $c_2 + c_3$. If $c_2 > 0$, then it is a stable spiral or a stable node. The orbits passing through this point contain profiles that oscillate (if $0 < c_2 + c_3 < 2$) or do not oscillate (if $c_2 + c_3 \geq 2$) with the behavior $f(\eta) \sim C_0 \eta^{-2/(1-m)}$ as $\eta \to \infty$. If $c_2 < 0$, the behavior is similar, but as $\eta \to \infty$ or as $\eta \to 0$, due to the different type of point we may have in this case. If instead of P_3 a limit cycle appears, this cycle contains the orbits oscillating in a strip bounded by two profiles with behavior $f_{1,2} \sim C_{1,2} \eta^{-2/(1-m)}$, both as $\eta \to \infty$ and as $\eta \to 0$.

• For the point Q_1, the analysis is similar to the previous cases, with the only difference that for $c_2 > 0$, the profiles behave like $f(\eta) \sim \eta^{-\alpha/\beta}$ as $\eta \to 0$ and for $c_2 < 0$ they have the same asymptotic behavior, as $\eta \to \infty$.

There are five possible models for the phase-plane. For $\beta > 0$, there is only one point where the trajectories enter, which is P_3, which can be a node or a spiral, and the model behavior is indicated in Figure 2.8a) in the case of a node (the other is similar). For $\beta < 0$, following the

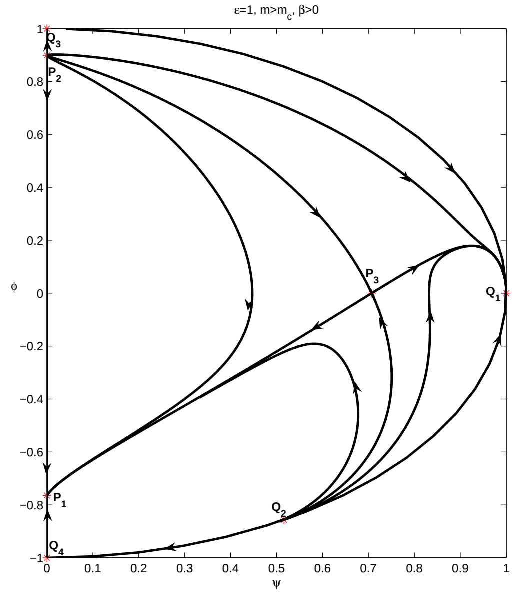

Figure 2.7: Phase portrait for Case 2. Experiment for $m = -0.7$, $n = 0.7$, $\beta = 1$.

analysis in Proposition 2.3 and taking into account that the behavior of the trajectories is determined by the connection between P_1 and Q_1, resp. by the union of these points with P_3 or with the limit cycle (when it exists), we have four models for the phase-plane, for $\beta \leq \beta_1^*$, $\beta_1^* < \beta < \beta_2^*$, $\beta = \beta_2^*$ and $\beta_2^* < \beta < 0$, indicated respectively in Figures 2.8b), 2.8c), 2.8d) and 2.8e).

Case 4: $\varepsilon = -1$ **and** $\mathrm{sgn}(b) = -1$. The point P_3 does not appear. The local analysis near the points P_1, P_2 and Q_2 is identical as in Case 2. The local analysis near the point Q_1 is similar as in Case 3. The trajectories can enter only in P_1, hence the phase-plane is topologically equivalent to the one in Figure 2.9.

Case 5: $\varepsilon = 0$ **and** $\mathrm{sgn}(b) \neq 0$. The local analysis near the points P_1, P_2 and Q_2 (which does not depend on ε) is identical as in Case 1 if $\mathrm{sgn}(b) = 1$ or as in Case 2, if $\mathrm{sgn}(b) = -1$, and the point P_3 does not appear. The unique point influenced by ε is Q_1. The center manifold of Q_1 has the form $V = o(W)$, hence $\Phi \to 0$ as $\eta \to \infty$. It follows that $2 + (1-m)\eta\varphi'/f \sim 0$, hence $f(\eta) \sim C\eta^{-2/(1-m)}$ as $\eta \to \infty$. Let us remark that, for $\varepsilon = 0$, this behavior is similar to that near Q_1 in the other cases, since $-\alpha/\beta = -2/(1-m)$ in this case.

The special case $b = 0$. In this case, as we have shown in detail in [78], the system changes into

$$\left\{ \begin{array}{l} \dot{\Psi} = \Psi\Phi, \\ \dot{\Phi} = c_1\Phi^2 - c_2\Psi\Phi - \Phi + \varepsilon\Psi, \end{array} \right. \tag{2.101}$$

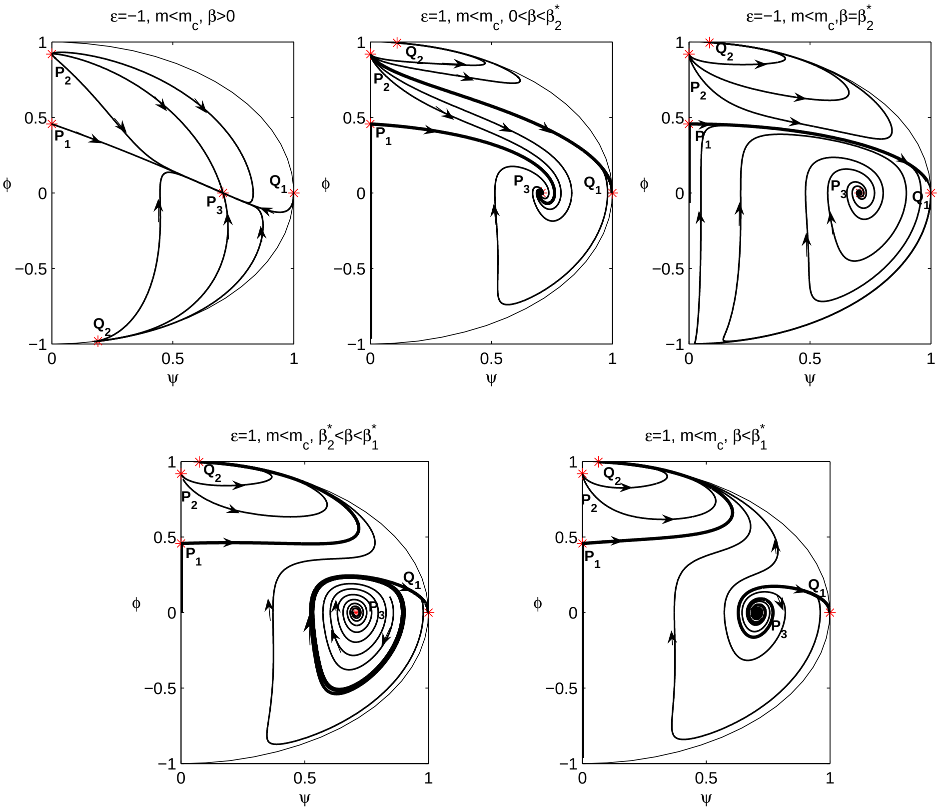

Figure 2.8: Phase portrait for Case 3. Experiment for $m = -5$, $n = 0.7$ and various values of β: (a) $\beta = 1$; (b) $\beta = -1.75$; (c)$\beta = \beta_2^*$; (d) $\beta = \beta_2^* - 0.35$; (e) $\beta = -3$.

with c_1 and c_2 as before. This system has only two critical points in the plane: $R_1 = (0, 0)$ and $R_2 = (0, (m-1)/m)$. The critical points at infinity are the same and with the same analysis as for (2.95).

• The linearized system near R_1 has eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$, hence this point is degenerate. With the methods in Section 2.12 of the book [109], we find that this point is a saddle-node, whose center manifold has the form $\Phi = \varepsilon\Psi + o(\Psi)$. Hence, in the region $\Psi > 0$, it behaves like a saddle if $\varepsilon = 1$ and like a stable node if $\varepsilon = -1$. We write $\Phi = \varepsilon\Psi + o(\Psi)$ in terms of profiles and integrate to obtain the behavior of f:

$$f(\eta) \sim \begin{cases} \eta^{2/(m-1)}\left(\frac{-\log\eta}{n-2}\right)^{1/(m-1)}, \text{ as } \eta \to \infty, \text{ if } \varepsilon = -1, \\ \eta^{2/(m-1)}\left(\frac{\log\eta}{n-2}\right)^{1/(m-1)}, \text{ as } \eta \to 0, \text{ if } \varepsilon = 1. \end{cases}$$

• The linearized system near R_2 has eigenvalues $\lambda_1 = (m-1)/m > 0$, $\lambda_2 = 1$, hence it is an

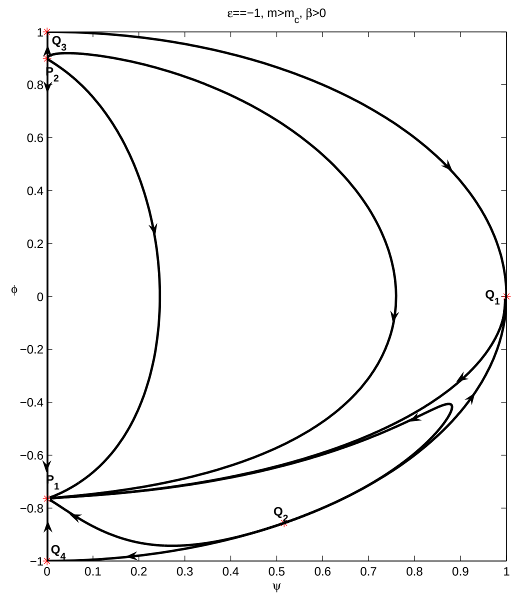

Figure 2.9: Phase portrait for Case 4. Experiment for $m = -0.7$, $n = 0.7$, $\beta = 1$.

unstable node. Since $\Phi \to (m-1)/m$ as $\eta \to 0$, we find that $f \sim C$ as $\eta \to 0$, hence R_2 acts like the point P_2 in the phase-plane of (2.95).

The special case $c_2 = 0$. This value of c_2 may affect the point P_3 in the plane and the points Q_1 and Q_2 are identified, let us denote by $R = (0,0)$ the new point. The linearization of the system (2.98) near R has only zero eigenvalues. In order to perform a local study, we set

$$S = -(1-c_1)V^2 - \varepsilon c_3 V\overline{W} + \overline{W} + \operatorname{sgn}(b)\overline{W}^2, \ \overline{W} = \varepsilon W$$

and transform the system into the "normal" form

$$\begin{cases} \dot{V} = S, \\ \dot{S} = (c_1-1)V^3(1+h(V)) + (2c_1-3)VS(1+g(V)) + S^2R(V,S), \end{cases}$$

which satisfies the conditions of Theorem 2, Section 2.11 of the book [109], with $m = n = 1$, $b_n = 2c_1 - 3 < 0$, $a_k = c_1 - 1 < 0$, $\lambda = ((m+1)/(m-1))^2 \geq 0$. It follows that R is a critical point with an elliptic domain, i. e. it has an elliptic sector, two parabolic sectors, a hyperbolic sector and two separatrices, cf. [109], pag. 148, where one can also see a picture of the phase portrait.

We pass to the local analysis in terms of profiles near R and we remark that a good approximation of the system (2.98) is realized by the following system:

$$\begin{cases} \dot{W} = -VW, \\ \dot{V} = -(1-c_1)V^2 + \varepsilon W, \end{cases}$$

which can be integrated explicitly to find

$$V = \pm\frac{\sqrt{-(m+1)[K(m+1)W^{2/(1-m)} + 2\varepsilon(m-1)W]}}{m+1}, \ K \in \mathbb{R}. \tag{2.102}$$

We divide the analysis of (2.102) in three cases, assuming for the moment that $\varepsilon \neq 0$:
(a) $m > -1$. If $\varepsilon = 1$, we are in the elliptic sector and we find $V \sim \sqrt{2(1-m)/(1+m)}W^{1/2}$. By integration we find $f(\eta) \sim (C - \sqrt{2(1-m)/(1+m)}\eta)^{2/(m-1)}$, $C > 0$, which has blow-up at some $\eta_0 < \infty$. If $\varepsilon = -1$, it is easy to check that the equality (2.102) is impossible.
(b) $m < -1$. If $K = 0$ in (2.102), there is only a solution for $\varepsilon = -1$, the same as in case (a). If $K \neq 0$, there are solutions both for $\varepsilon = 1$ and $\varepsilon = -1$, with the local behavior $f(\eta) \sim (C - (1-m)\eta^{2m/(m-1)})^{1/m}$. There are also solutions entering R, in both cases (a) and (b), which behave like a constant C as $\eta \to 0$.
(c) $m = -1$. Then the relation (2.102) is no longer valid; instead, we have

$$V = \pm\sqrt{(K - 2\varepsilon\log(W))W}.$$

By integration, we obtain the profiles f satisfying $f^{'}(\eta) \sim Cf(\eta)^{(3-m)/2}(2\log(\eta) + (1-m)\log(f(\eta)))^{1/2}$, hence blowing-up at some $\eta_0 < \infty$.

In conclusion, at least for $\varepsilon \neq 0$, this point contains only blow-up profiles. This result is natural if we think that R is the union of the "old" points Q_1 and Q_2, that Q_2 was the blow-up point in the standard case and that the behavior of the profiles entering Q_1, which is $\eta^{-\alpha/\beta}$, also blows-up when $\beta = 0$.

The special case $n = 2$. The unique difference is that in this case the points P_1 and P_2 coincide, set $P = (0, \sqrt{(m-1)/m})$ this point, which is now a saddle-node. By standard analysis, we find that all the orbits entering or coming from P have the behavior $f(\eta) \sim C$ as $\eta \to 0$ or $\eta \to \infty$, which is the common behavior near the old points P_1 and P_2.

Dimension $n > 2$. To pass to dimensions $n > 2$, we use the self-maps introduced in Section 2.10, where always the subscripts 1 and 2 will refer to profiles and variables in dimension $n_1 \in [2, \infty)$, resp. $n_2 \in (2/(1-m), 2]$. Moreover, using the results in Theorem 2.3, we find the relations between the exponents:

$$\beta_1 = \frac{mn_2 - n_2 + 2}{2m}\beta_2.$$

Using these facts and direct calculations, we establish the following correspondences between the possible rates at $\eta \to 0$ or at $\eta \to \infty$ of the profiles:
(a) If $f_2(\eta_2) \sim C_0\eta_2^{-2/(1-m)}$, then $f_1(\eta_1) \sim C_0\eta_1^{-2/(1-m)}$. In the same way, the profiles having a logarithmic correction transform into profiles with the same logarithmic correction $f_1(\eta_1) \sim C\eta_1^{\frac{2}{m-1}}\log(\eta_1)^{\frac{1}{m-1}}$.
(b) If $f_2(\eta_2) \sim C\eta_2^{-\alpha_2/\beta_2}$, then $f_1(\eta_1) \sim C\eta_1^{-\alpha_1/\beta_1}$.
(c) If $f_2(\eta_2) \sim C\eta_2^{(2-n_2)/m}$, then $f_1(\eta_1) \sim C$; on the other hand, if $f_2(\eta_2) \sim C$, then $f_1(\eta_1) \sim C\eta_1^{(2-n_1)/m}$. Hence the self-map interchanges these two different rates. Recall that a similar effect of the self-map holds also in the case $m > 0$, cf. Section 2.6.

(d) Since in the range where the self-maps hold, $n_2 \in [2/(1-m), 2)$, we always have $m < m_c$, it is easy to check that the blow-up profiles are mapped into blow-up profiles and the large solutions (in the sense that f_2 is defined in (η_0, ∞) and explodes as $\eta \to \eta_0$, $\eta > \eta_0 > 0$) are mapped into large solutions.

2.12.3 Global behavior of self-similar solutions

We gather under the form of three theorems (for self-similarities of type I, II and III) the results obtained in the previous subsection, specifying the situations in which each local rate appears. Recall that the cases $m > m_c$ and $m = m_c$ are possible in our range only for dimensions $n < 2$.

Forward self-similarity ($\varepsilon = 1$). We have the following:

Theorem 2.10. *(a) If $m < m_c$, all the profiles with $\beta > 0$ satisfy $f(\eta) \sim C\eta^{-\alpha/\beta}$ as $\eta \to \infty$. For $\eta \sim 0$, for any $C > 0$, there exists a unique profile f such that $f(0) = C$, $f'(0) = 0$. Moreover, there exists a unique one-parameter family of profiles such that $f(\eta) \sim C\eta^{(2-n)/m}$, and all the other profiles are large solutions defined on (η_0, ∞) for some $\eta_0 > 0$. These two families of profiles coincide if $n = 2$. All the profiles with $\beta < 0$ blow-up as $\eta \to \eta_0$ with $\eta < \eta_0 < \infty$.*
(b) If $m > m_c$ and $n < 2$, there exist profiles providing all possible combinations of different *rates as $\eta \to \infty$ and $\eta \to 0$, except for the explicit solution $f(\eta) = C_0\eta^{-2/(1-m)}$, $C_0^{m-1} = (1-m)/(2(mn-n+2))$.*
(c) If $m = m_c$ and $n < 2$, all the profiles behave as in part (a), with the exception of the behavior $f(\eta) \sim C\eta^{(2-n)/m}$, which in this case converts into $f(\eta) \sim (\eta\sqrt{\log\eta/(n-2)})^{2/(m-1)}$ as $\eta \to 0$.

Backward self-similarity ($\varepsilon = -1$). In this case the analysis is larger and more involved, due to the appearance of the limit cycle in the phase-plane. Let us consider first the two critical exponents and the explicit constant (for $m < m_c$):

$$\beta_1^* = -\frac{(n+2)(m-m_s)}{2(mn-n+2)}, \ \beta_2^* = \frac{-m}{mn-n+2}, \text{ if } n < 2, \ \beta_2^* = -\frac{1}{2}, \text{ if } n \geq 2,$$
$$C_0 = \left(\frac{m-1}{2(mn-n+2)}\right)^{\frac{1}{m-1}}.$$

Theorem 2.11. *(a) If $m < m_c$, the results change with β:*

- *If $\beta \leq \beta_1^*$, there exists a unique one-parameter family with $f(\eta) \sim C\eta^{-\alpha/\beta}$ as $\eta \to \infty$ and $f(\eta) \sim C_0\eta^{-2/(1-m)}$ as $\eta \to 0$. All the other profiles blow-up as $\eta \to \eta_0$, $\eta < \eta_0 < \infty$.*
- *If $\beta \in (\beta_1^*, \beta_2^*)$, all profiles behaving as in the previous case still exist. Moreover, three new families of profiles appear: a one-parameter family of profiles with $f(\eta) \sim C\eta^{-\alpha/\beta}$ as $\eta \to \infty$, another family with $f(\eta) \sim C_0\eta^{-2/(1-m)}$ as $\eta \to \infty$ and another family of profiles oscillating as $\eta \to \infty$ in a strip between $f_1(\eta) = C_1\eta^{-2/(1-m)}$ and $f_2(\eta) = C_2\eta^{-2/(1-m)}$, with $0 < C_1 < C_0 < C_2$. All these new families oscillate in a similar strip as $\eta \to 0$.*

• *If $\beta = \beta_2^*$, there is an explicit one-parameter family $f(\eta) = C\eta^{(2-n)/m}$, and a family with $f(\eta) \sim C_0\eta^{-2/(1-m)}$ and oscillating as $\eta \to 0$. All the other profiles blow-up.*

• *If $\beta \in (\beta_2^*, 0)$ and $n < 2$, there exists a unique one-parameter family of profiles with $f(\eta) \sim \eta^{-\alpha/\beta}$ as $\eta \to \infty$ and $f(0) = C > 0$, $f'(0) = 0$, and another unique one-parameter family with $f(\eta) \sim C_0\eta^{-2/(1-m)}$ as $\eta \to \infty$ and $f(\eta) \sim C\eta^{(2-n)/m}$ as $\eta \to 0$. For $n \geq 2$, these two families interchange the behavior at $\eta = 0$. Moreover, there are profiles with $f(\eta) \sim C_0\eta^{-2/(1-m)}$ as $\eta \to \infty$ and $f(0) = C > 0$, $f'(0) = 0$, if $n < 2$, or $f(\eta) \sim C\eta^{(2-n)/m}$ if $n \geq 2$. All the other profiles blow-up.*

• *If $\beta \geq 0$, all the profiles satisfy $f(\eta) \sim C_0\eta^{-2/(1-m)}$ as $\eta \to \infty$ and there are profiles with all the possible rates as $\eta \to 0$.*

(b) If $m > m_c$ and $n < 2$, the rate $\eta^{-2/(1-m)}$ disappears. All the profiles with $\beta > 0$ may have all the other possible rates, except from $\eta^{(2-n)/m}$, as $\eta \to 0$ and satisfy $f(\eta) \sim C\eta^{(2-n)/m}$ as $\eta \to \infty$. For $\beta < 0$, the profiles may provide all the other possible rates as $\eta \to \infty$ and will all satisfy $f(0) = C$, $f'(0) = 0$.

(c) If $m = m_c$ and $n < 2$, all the behaviors are as in Theorem 2.10, part (c), and it appears a family with $f(\eta) \sim (\eta\sqrt{\log \eta/(2-n)})^{-2/(1-m)}$ as $\eta \to \infty$.

As an idea of proof, we start from the local analysis in terms of profiles performed in Section 2.12 and we translate into profiles all the possible connections in the phase-plane (see Figure 2.8 and Figure 2.9). In particular, the critical values β_1^* and β_2^* are the exponents which delimitate the situation where a limit cycle appears (cf. Proposition 2.3).

Exponential self-similarity ($\varepsilon = 0$). In this case, the rates $\eta^{-\alpha/\beta}$ and $\eta^{-2/(1-m)}$ coincide and again logarithmic corrections may appear.

Theorem 2.12. *(a) If $m < m_c$, the analysis is similar to that of Theorem 2.10, part (a). If $m > m_c$, the analysis is similar to that of Theorem 2.11, part (b).*

(b) If $m = m_c$, there exists an explicit two-parameter family of profiles $f(\eta) = (C + (1-m)\beta\eta^2/2)^{1/(m-1)}$. All the other profiles which do not blow-up satisfy $f(0) = C$, $f'(0) = 0$ and $f(\eta) \sim (\eta\sqrt{\log \eta})^{2/(m-1)}$ as $\eta \to \infty$. We analyze in more detail this case in Subsection 2.12.4, part (e).

2.12.4 Special and explicit profiles

(a) We first look for *lines in the phase-plane*, i. e. solutions which satisfy $\Phi = a_1\Psi + a_2$ for some a_1, a_2. Looking for lines going out of P_2, we find the Barenblatt profile:

$$f(\eta) = \Big(C - \frac{m-1}{2(mn-n+2)}\eta^2\Big)^{1/(m-1)}, \ \beta = 1/(mn-n+2). \tag{2.103}$$

If $m > m_c$, then $k = (m-1)/2(mn-n+2) < 0$. Thus, the Barenblatt profiles are solutions of type I for $m > m_c$. If $m < m_c$, the profile (2.103), is a blow-up profile (of type I again). We also find the trivial profile $f \equiv C$, with $\alpha = 0$, $\beta = \pm 1/2$.

Looking for lines going out of P_1, we find the dipole profile:

$$f(\eta) = \eta^{(2-n)/m}\Big(C - \frac{m-1}{2(mn-n+2)}\eta^{(mn-n+2)/m}\Big)^{1/(m-1)}, \ \beta = 1/2m. \tag{2.104}$$

which exists for $m > m_c$ and it transforms into a blow-up profile as passing through $m = m_c$. We also find the profile $f(\eta) = C\eta^{(2-n)/m}$, giving rise to the stationary solution $u(x,t) = C|x|^{(2-n)/m}$.

(b) We look now for an explicit profile of the form $f(\eta) = C\eta^{-2/(1-m)}$. We verify the equation (2.24) and we find that $C^{m-1} = -(m-1)\varepsilon/(2(mn-n+2))$. For $m > m_c$, such a solution exists for $\varepsilon = 1$ and is of type I. For $m < m_c$, the solution exists if $\varepsilon = -1$ and is of type II. In the phase-plane, these profiles are represented by the point P_3 itself.

(c) The *Loewner-Nirenberg type solution.* This is a special solution introduced by Loewner and Nirenberg for the usual fast-diffusion case, $0 < m < 1$, useful in applications in differential geometry, cf. [78] or [127]. It exists for $m = m_s := (n-2)/(n+2)$, $\beta = 0$ and $\varepsilon = -1$. Its analogue for the VFDE can exist only for $n < 2$; we introduce in the system (2.95) $c_2 = c_3 = 0$ and we integrate to find

$$f(\eta) = \left(C + \frac{\eta^2}{4nC}\right)^{-(n+2)/2}, \tag{2.105}$$

which is the same formula as for the usual fast-diffusion case. For $\varepsilon = 1$, we find again a Loewner-Nirenberg type solution, $f(\eta) = ((\eta^2 + K)/\sqrt{nK})^{-(n+2)/2}$, with $K > 0$.

(d) We look for explicit solutions with $b = 0$, $\varepsilon = 0$, i.e. solutions of type III for $m = m_c$. By integrating (2.95), we obtain

$$\Phi = \frac{c_2}{c_1 - 1}\Psi + \frac{1}{c_1} + K\Psi^{c_1}, \ K \in \mathbb{R}. \tag{2.106}$$

If $K = 0$, we can integrate explicitly the equation above in terms of profiles and we find that

$$f(\eta) = \left(C + \frac{(1-m)\beta}{2}\eta^2\right)^{1/(m-1)}, \tag{2.107}$$

where the exponent β changes with the explicit profile and $C > 0$. We remark that these profiles can be seen as limits as $m \to m_c$ of the Barenblatt profiles (2.103). If we look for solutions with $K \neq 0$ in (2.106), we observe that for $n < 2$, the behavior of the resulting ODE can be well approximated as $\Psi \to \infty$ by the ODE $\Phi = c_2\Psi/(c_1 - 1)$, hence $f(\eta) \sim (\eta\sqrt{\log \eta})^{2/(m-1)}$ as $\eta \to \infty$. We end in this way the proof of Theorem 2.12, part (c).

(e) Other solutions with $\beta = 0$. If we look for a power-like behavior of the phase-plane variables in (2.98) near the critical point $R = (0,0)$, it is easy to check that we should necessarily have

$$V = \sqrt{\frac{2(1-m)}{|1+m|}}(\pm W^{1/2} + W),$$

and, translating in terms of profiles, we obtain the explicit solution:

$$f(\eta) = \left(\frac{2|1+m|}{1-m}\right)^{1/(1-m)} (\eta - \eta_0)^{-2/(1-m)}, \ n = 1.$$

This profile belongs to a solution of type II for $m > -1$ and to a solution of type I for $m < -1$ and also appears in [60]. If we want that $\varepsilon = 0$ and $\beta = 0$, we obtain a family of solutions

of type III whose profiles are: $f(\eta) = (C_1\eta^{2-n} + C_2)^{1/m}$, C_1, $C_2 \in \mathbb{R}$. Finally, for the case $m = -1$ and $\varepsilon = 0$ we obtain the trivial type III solution $f(\eta) = 1/(C_1 + C_2\eta)$, with C_1, $C_2 \in \mathbb{R}$.
(f) If $n \geq 2$, everything that holds for $m < m_c$ in the previous analysis remains unchanged, except from the dipole profile, whose singularity at $\eta = 0$ disappears.

2.13 Self-similar solutions for the very fast p-Laplacian equation

We have seen (in Theorem 2.8) that in order to study the self-similarity of the FPLE, it is enough to start from the self-similarity for the VFDE in dimensions $0 < n < 2$, performed in detail in the previous section. Recall also that self-similar solutions of the FPLE solve:

$$\frac{1}{p-1}\eta^{1-n}(\eta^{n-1}|f'|^{p-2}f')' + \alpha f + \beta\eta f' = 0. \tag{2.108}$$

2.13.1 The FPLE with $p \neq 0$

We mainly use the transformations indicated in Theorem 2.2. Nevertheless, it is also useful to consider a second correspondence relation between self-similar solutions, deduced in [78]:

$$\bar{f}(\bar{\eta}) = -\frac{mn-n+2}{m(m+1)}\eta^{n-1}\frac{D_1}{\bar{\alpha}}\left(|f(\eta)|^{m-1}f^{'}(\eta) + m\beta\eta f(\eta)\right), \tag{2.109}$$

where $\bar{\alpha} \neq 0$ (for the case $\bar{\alpha} = 0$ see Section 2.5), which we use to identify precise constants.
(a) If $f(\eta) \sim C\eta^{(2-n)/m}$, we find that $\bar{f}(\bar{\eta}) \sim C_1\bar{\eta}^{p/(p-1)} + C_2$, as $\bar{\eta} \to 0$ or $\bar{\eta} \to \infty$, where C_2 is a constant which is not free, but depends on C_1. In particular, this last constant may be also 0, and in this case it follows that $\bar{f}^{'}$ blows-up at $\bar{\eta} = 0$.
(b) If $f(\eta) \sim C\eta^{-\alpha/\beta}$, then, recalling the relations between exponents $\bar{\beta} = (mn-n+2)\beta/(m+1)$, $\bar{\alpha} = \big[(mn-n+2)\alpha - n\varepsilon\big]/2$ (cf. Theorem 2.3), we find $\bar{f}(\bar{\eta}) \sim C_1\bar{\eta}^{-\bar{\alpha}/\bar{\beta}}$, as $\bar{\eta} \to 0$ or $\bar{\eta} \to \infty$.
(c) If $f(\eta) \sim C_0\eta^{-2/(1-m)}$, we transform also C_0 and obtain $\bar{f}(\bar{\eta}) \sim \bar{C}_0\bar{\eta}^{-p/(2-p)}$, with the precise constant $\bar{C}_0 = ((p-2)\bar{n} + p)/(2(p-1)))^{1/(p-2)}$, as $\bar{\eta} \to 0$ or $\bar{\eta} \to \infty$.
(d) If $f(0) = C > 0$ and $f^{'}(0) = 0$, then $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{(p-\bar{n})/(p-1)}$ as $\bar{\eta} \to 0$. On the other hand, the profiles of the VFDE with $f(\eta) = C_1 + C_2\eta^{2-n} + o(\eta)$ as $\eta \to 0$ transform into profiles of the VFDE with $\bar{f}(\bar{\eta}) \sim K + C\bar{\eta}^{(p-\bar{n})/(p-1)}$, as $\bar{\eta} \to 0$. Here, on the contrary of case (a), the constants K and C are independent.
(e) The blow-up or large profiles f of the VFDE with vertical asymptote in an intermediate point $\eta_0 \in (0,\infty)$ are mapped into profiles of the FPLE which present the phenomenon of *quenching*, i. e. blow-up of the derivative (gradient) of the profile. These profiles of the FPLE do not blow-up: if the blow-up rate of f is $f(\eta) \sim C(\eta-\eta_0)^{1/(m-1)}$, then near $\bar{\eta}_0$ we have $\bar{f}(\bar{\eta}) \sim (\bar{\eta}-\bar{\eta}_0)^{(p-1)/(p-2)}$. We will remark the same phenomena on explicit examples below.

(f) The profiles of VFDE with logarithmic corrections in the behavior transform into profiles of the FPLE presenting again logarithmic rate. If $f(\eta) \sim C(\eta\sqrt{\log\eta})^{2/(m-1)}$, we find $\bar{f}(\bar{\eta}) \sim C(\bar{\eta}^p \log\bar{\eta})^{1/(p-2)} + C_2$.

We also remark that $p > p_c := 2\bar{n}/(\bar{n}+1)$ can hold in our range only if $\bar{n} < 1$, hence the interesting case is always $p < p_c$, obtained through mappings in Theorem 2.8, parts (a) and (b), both from $m > -1$, $m > m_c$ and from $m < -1$, $m < m_c$. The critical exponents in Theorem 2.11 are now

$$\bar{\beta}_1^* = -\frac{(p-1)((p-2)\bar{n}+2p)}{p((p-2)\bar{n}+p)}, \ \bar{\beta}_2^* = -\frac{p-1}{p},$$

satisfying $\bar{\beta}_1^* < \bar{\beta}_2^* < 0$, the same order as in the VFDE case.

With these transformations, the general results in the case $p \in (0,1)$ are obtained copying line by line parts (b) of the statements of Theorems 2.10, 2.11 and 2.12 and changing the rates of behavior according to the list above. The general results for $p < 0$ are obtained copying line by line parts (a) of the same theorems. We leave this simple task to the reader.

Some explicit solutions.

(i) Starting from the dipole solution (2.104) of the VFDE, we integrate and obtain the form of the Barenblatt solution for the FPLE,

$$\bar{f}(\bar{\eta}) = -\frac{4(p-1)}{p+\bar{n}(p-2)}(C - k\bar{\eta}^{p/(p-1)})^{(p-1)/(p-2)}, \ k = \frac{(p-2)(\bar{n}(p-2)+p)}{4p(p-1)}. \tag{2.110}$$

This is an explicit example of a profile with no blow-up, but presenting the phenomenon of quenching at some point in $(0,\infty)$.

(ii) In a similar manner, starting from the Barenblatt profile of the VFDE (2.103), we find a special profile with quenching (obtained also in Section 2.5), whose derivative is

$$\bar{f}'(\bar{\eta}) = \bar{\eta}^{(1-\bar{n})(p-1)}\left(C - \frac{(p-2)(p-1)}{p(\bar{n}(p-2)+p)}\bar{\eta}^{(\bar{n}(p-2)+p)/(p-1)}\right)^{1/(p-2)}. \tag{2.111}$$

(iii) There is an explicit profile of the form $\bar{f}(\bar{\eta}) = \bar{C}_0\bar{\eta}^{-p/(2-p)}$, of type II. Integrating the constant profile with $\alpha = 0$, $\beta = \pm 1/2$ in the VFDE case, we obtain $\bar{f}(\bar{\eta}) = C\bar{\eta}^{(p-\bar{n})/(p-1)}$, having self-similarity exponents $\bar{\alpha} = (\bar{n}-p)\varepsilon/(\bar{n}(p-2)+p)$ and $\bar{\beta} = \pm(p-1)/(\bar{n}(p-2)+p)$.

(iv) The Loewner-Nirenberg profile for the FPLE has the expression

$$\bar{f}(\bar{\eta}; C) = \frac{4n}{2-n}\left(\frac{4n^3C}{(2-n)^2}\right)^{\frac{n-2}{4}}\left(1 + C\bar{\eta}^{\frac{2n}{n-2}}\right)^{-\frac{n}{2}}, \tag{2.112}$$

in dimension $\bar{n} = n$, coming from the similar Loewner-Nirenberg profile of the VFDE in the same dimension n. These profiles appear for $p = p_s = 2\bar{n}/(\bar{n}+2)$, which in our range only can hold for $0 < \bar{n} < 2$.

(v) There exists also a solution of type III, for $p = p_c$, which can be seen as limit of the Barenblatt solutions as $p \to p_c$. Its derivative is

$$\bar{f}'(\bar{\eta}) = \bar{\eta}^{-(\bar{n}-1)/(p-1)}\left(C + \frac{(2-p)(\bar{n}(p-2)+p)\bar{\beta}}{4(p-1)}\bar{\eta}^{((p-2)\bar{n}+p)/(p-1)}\right)^{1/(p-2)}.$$

If we look for solutions of type III with $\bar{\beta} = 0$, by direct integration in (2.108), we find the family $\bar{f}(\bar{\eta}) = C_1 \bar{\eta}^{(p-\bar{n})/(p-1)} + C_2$.

2.13.2 The special case $p = 0$

We use the transformations obtained in Section 2.11, which, in terms of profiles of self-similar solutions, become

$$\bar{f}'(\bar{\eta}) = \frac{1}{\bar{n}} \eta^{1-1/\bar{n}} f(\eta), \ \bar{\eta} = \eta^{1/\bar{n}}, \ \bar{\beta} = \frac{1}{\bar{n}} \beta,$$

where the last relation is obtained using also (2.108). We also have to take into account that we start from profiles of the VFDE with $m = m_c = -1$ in dimension $n = 1$. With these transformations and starting from part (c) of Theorems 2.10, 2.11 and 2.12, we gather the results for the self-similarity of the FPLE with $p = 0$ in the following

Theorem 2.13. *(i) All the profiles with $\varepsilon = 1$ (i. e. of type I) with $\beta > 0$ satisfy $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{1/2\bar{n}}$ as $\bar{\eta} \to \infty$. For any $C > 0$, there exists a unique profile such that $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{\bar{n}}$ as $\bar{\eta} \to 0$. Moreover, there exists a unique one-parameter family of profiles with $\bar{f}(\bar{\eta}) \sim C\sqrt{|\log \bar{\eta}|}$ as $\bar{\eta} \to 0$. All the other profiles with $\bar{\beta} > 0$, together with all the profiles with $\bar{\beta} < 0$, present quenching in some point $\bar{\eta}_0 \in (0, \infty)$, with precise quenching rate $\bar{f}(\bar{\eta}) \sim C(\bar{\eta} - \bar{\eta}_0)^{1/2}$.*
(ii) The profiles with $\varepsilon = -1$ (i. e. of type II) are classified similarly as in part (a), with the unique difference that the profile with $\bar{f}(\bar{\eta}) \sim C\sqrt{\log \bar{\eta}}$ appears as $\bar{\eta} \to \infty$.
(iii) The profiles with $\varepsilon = 0$ (i. e. of type III) have $\bar{\alpha} = 0$. There is a special family of profiles of the FPLE with the explicit expression

$$\bar{f}(\bar{\eta}) = \frac{1}{\bar{n}^2 \sqrt{\beta}} \log \left(\bar{\eta}^{\bar{n}} \sqrt{\beta} + \sqrt{C + \beta \bar{\eta}^{2\bar{n}}} \right) + K, \tag{2.113}$$

All the other profiles without quenching satisfy $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{\bar{n}}$ as $\eta \to 0$ and $\bar{f}(\bar{\eta}) \sim C\sqrt{\log \bar{\eta}} + C_2$ as $\bar{\eta} \to \infty$.

Remark. In particular, we deduce that, for $\bar{n} > 1$, the FPLE with $p = 0$ admits profiles with both $\bar{f}(0) = 0$, $\bar{f}'(0) = 0$.

Some explicit profiles. We already have an explicit family of profiles, given by (2.113). We present other explicit profiles in what follows:
(a) Starting from $f(\eta) = C$, we find the family $\bar{f}(\bar{\eta}) = C\bar{\eta}^{\bar{n}}$, having $\bar{\alpha} = -\varepsilon/2$ and $\bar{\beta} = \pm 1/2\bar{n}$.
(b) Looking for solutions of the form $\bar{f}(\bar{\eta}) = C\sqrt{|\log \bar{\eta}|}$, we find the profile $\bar{f}(\bar{\eta}) = \pm 2\sqrt{\bar{n} \log \bar{\eta}}$, with $\bar{\beta} = -1/2\bar{n}$, $\bar{\alpha} = -1/2$ and of type I, presenting quenching at $\bar{\eta} = 1$. On the other hand, there exists another explicit profile with the same expression defined for $0 < \bar{\eta} < 1$, having $\bar{\beta} = 1/2\bar{n}$ and quenching at $\bar{\eta} = 1$. Every family of profiles with quenching at some $\bar{\eta}_0 \in (0, \infty)$ appears as pairs of profiles as in this case.
(c) There exists an explicit family of profiles of type III, given by $\bar{f}(\bar{\eta}) = C_1 \bar{\eta}^{\bar{n}} + C_2$, with $\varepsilon = 0$ and $\bar{\alpha} = \bar{\beta} = 0$. This is obtained by direct integration.

2.14 Study of integrability for FPLE and proof of Theorem 2.9

We check the integrability of every possible rate from the list above, both at $\bar{\eta} = 0$ and infinity.

(i) The profiles with $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{p/(p-1)} + K$ can be integrable at infinity for $K = 0$, and we need to have $p/(p-1) < -\bar{n}$, i.e. $p \in (\bar{n}/(1+\bar{n}), 1)$. The resulting solution is integrable near the origin in the complementary case $p \in (0, \bar{n}/(1+\bar{n}))$ and also for any $p < 0$.

(ii) The profiles with $\bar{f}(\bar{\eta}) \sim \bar{C}_0\bar{\eta}^{-p/(2-p)}$ give solutions which are never integrable near infinity and always integrable near $\bar{\eta} = 0$. This follows by the fact that $p < p_c$, for any $\bar{n} \geq 1$, in our range.

(iii) The profiles with $\bar{f}(\bar{\eta}) \sim K + C\bar{\eta}^{(p-\bar{n})/(p-1)}$ as $\bar{\eta} \to 0$ give always solutions which are integrable near $x = 0$.

(iv) For the profiles $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{-\bar{\alpha}/\bar{\beta}}$, the integrability condition at infinity transforms into $\left[\varepsilon - \bar{\beta}(\bar{n}(p-2)+p)\right]/\bar{\beta} < 0$. Taking into account that always $\bar{n}(p-2)+p < 0$ (since $p < p_c$), for $\bar{\beta} > 0$ we find that necessarily $\varepsilon = -1$ and $\bar{\beta} \in (0, -1/(\bar{n}(p-2)+p))$. If $\bar{\beta} < 0$, we find that $\varepsilon = 1$ and $\bar{\beta} \in (1/(\bar{n}(p-2)+p), 0)$.

We study the possible connections of points that give finite mass solutions. We recall (cf. [78]) that the phase-plane is the same for both equations and topologically equivalent with those given in Figures 2.6, 2.7, 2.8 and 2.9, only that $m > -1$, $m > m_c$ transforms into $p > 0$ and $m < -1$, $m < m_c$ transforms into $p < 0$. For $\varepsilon = 1$ and $p < 0$, all the profiles with $\bar{\beta} > 0$ go to infinity like $\bar{\eta}^{-\bar{\alpha}/\bar{\beta}}$ and behave near $\bar{\eta} = 0$ in all the other possible ways. We thus obtain finite mass solutions only for $\bar{\beta} \in (0, -1/(\bar{n}(p-2)+p))$, in particular these solutions do not conserve mass. For $\bar{\beta} < 0$ all the profiles have quenching.

If $\varepsilon = 1$ and $0 < p < 1$, there are finite mass solutions as above, but if $p \in (\bar{n}/(1+\bar{n}), 1)$ there are other finite mass solutions, with behavior as in (i) above. Among them, there exists a *solution which conserve mass*, having the exponent $\bar{\beta} = 1/(\bar{n}(p-2)+p)$ and connecting in the phase-plane the points P_3 and P_1 for this value of $\bar{\beta}$. Consequently, it behaves like $\bar{\eta}^{p/(p-1)}$ as $C\bar{\eta} \to \infty$ and like $\bar{C}_0\bar{\eta}^{-p/(2-p)}$ as $\bar{\eta} \to 0$. We remark that this solution is different from the Barenblatt profile and not explicit, but it can be seen in some sense as a "reversed Barenblatt", taking into account its evolution in time.

If $\varepsilon = -1$ and $p < 0$, the only profiles that can induce finite mass solutions are those with $\bar{\beta} < 0$ and $\bar{f}(\bar{\eta}) \sim C\bar{\eta}^{-\bar{\alpha}/\bar{\beta}}$ as $\bar{\eta} \to \infty$. The integrability imposes $\bar{\beta} > 1/(\bar{n}(p-2)+p) > \bar{\beta}_2^*$. From the detailed phase-plane analysis performed in Section 2.12, in this range there exists only one orbit of profiles of this type, connecting P_1 and Q_1 (see Figure 2.8b). But a more careful study shows that this connection is unique and has $\bar{\beta} = -(p-1)/(\bar{n}(p-2)+p) < 1/(\bar{n}(p-2)+p)$ for $p < 0$, hence this has not finite mass. Consequently, there are no finite mass solutions for this range of p.

If $\varepsilon = -1$, $0 < p < 1$, by a similar discussion as above, there exist solutions with finite mass for any $p \in (\bar{n}/(1+\bar{n}), 1)$, whose profiles can be seen in the phase-plane as connecting P_2 with P_1 (see Figure 2.9) and behaving near $x = 0$ as in part (iii) above and near infinity as in part (i) above. In particular, there exists an orbit of solutions of this type with $\bar{\beta} =$

$-1/(\bar{n}(p-2)+p)$, which *conserve mass until extinction time*, having $f(0)=K>0$.

Finally, since the profiles with logarithmic corrections are never integrable, the case $p=0$ can be seen as a continuation of the study performed above.

2.15 An application: level-connecting profiles of the FPLE

We end the present chapter with a section dedicated to a more specific application of the FPLE: we look for profiles that we refer for short as *level connecting*. By this term we understand the profiles starting from a constant level (at $\bar{\eta}=0$ or by quenching at a finite point) and behaving like a constant in the end (as $\bar{\eta}\to\infty$ or with quenching at a finite point). These profiles are of interest in problems of image processing, as discussed in [14]. More precisely, these solutions appear in a natural manner in techniques of *contour enhancement*: if we think on the function $u(x,y)$ representing the grey level of the image at each point, then $0\le u\le C$ and u satisfies a nonlinear degenerate parabolic equation of the FPLE type, depending on the chosen model. As showed in [14] and [13], a realistic model of image enhancement involves the two contour-conditions: that $u=0$ on the left-hand side of the contour and $u=C$ on the right-hand side of the contour. On the other hand, other models involve the appearance of a singular free boundary problem, where the free boundary appears due to a blow-up in the gradient, that is our quenching phenomenon.

We emphasize on the profiles with quenching and with $\bar{\alpha}=0$ (which are solutions of the singular free boundary problem, as described in Section 6 of [14]). Since we deal only with FPLE, we drop the notation with bar.

Profiles with $\alpha=0$. In the case $\alpha=0$, the previous considerations based on the phase-plane variables are no longer valid, but the equation (2.108) is integrable. Indeed, if we put $g=\eta^{n-1}|f'|^{p-2}f'$, then g satisfies the equation

$$\frac{1}{p-1}g'+\beta\eta^{(np-2n+1)/(p-1)}g^{1/(p-1)}=0,$$

which, after integration, gives the derivative of f:

$$f'=\eta^{\frac{1-n}{p-1}}\left(K+\frac{(2-p)(p-1)\varepsilon}{p(np-2n+p)}\eta^{\frac{np-2n+p}{p-1}}\right)^{\frac{1}{p-2}},$$

where $\beta=\varepsilon/p$. Here, $K\in\mathbb{R}$ is an integration constant. We discuss the behavior and type of this solution with respect to ε and p. Note also that $f+C$ is another good self-similar profile in this case.

(a) If $p<0$ and $\varepsilon=1$, or $p>0$ and $\varepsilon=-1$, then we have to restrict K to be positive. We obtain a family of profiles with $f(\eta)\sim C+\eta^{(p-n)/(p-1)}$ as $\eta\to0$ and f presents quenching in a finite point depending on K. Let us remark that, given two constant levels, there exists a profile in this family connecting them. Indeed, if we fix $f(0)=C_1>0$ and the final level to be C_2, then we can choose the two independent constants (K and the translation) in order to satisfy these conditions. Note also that in the case $\varepsilon=1$ and $p<0$, we reobtain the profile (2.111).

(b) If $p > 0$ and $\varepsilon = 1$, or $p < 0$ and $\varepsilon = -1$, then $\beta < 0$, hence we can take any constant K. The behavior of the profile depends on the choice of K as follows:

• If $K > 0$, we obtain profiles with $f(\eta) \sim C + \eta^{(p-n)/(p-1)}$ as $\eta \to 0$ and $f(\eta) \sim C + \eta^{p/(p-2)}$ as $\eta \to \infty$. Note that these profiles are level-connecting only for $p > 0$ (and $\varepsilon = 1$).

• If $K < 0$, the profiles start from a quenching point, and behave like $f(\eta) \sim C + \eta^{p/(p-2)}$ as $\eta \to \infty$. They connect constant levels for $p > 0$ and $\varepsilon = 1$. Let us remark again that, fixing any two levels $C_1 \in (0, \infty)$ and $\lim_{\eta\to\infty} f(\eta) = C_2$ as limit (contour) conditions, there exists a profile in the family satisfying them.

• If $K = 0$, we find the explicit profile

$$f(\eta) = C\eta^{p/(p-2)} + C_0, \; C = \frac{p-2}{p}\left(\frac{\varepsilon(2-p)(p-1)}{p(np-2n+p)}\right)^{\frac{1}{p-2}}.$$

Other level-connecting profiles. There are many other level-connecting profiles, with $\alpha \neq 0$. These profiles correspond in the phase-plane to unions of the following points: $P_1 - Q_2$, $P_2 - Q_2$, $P_1 - P_2$. It is easy to check in the phase-plane when these connection may appear. The difference with respect to the profiles with $\alpha = 0$ is that, in this case, the levels that are connected evolve with time. As an interesting example, we have proved in Section 2.14 that there exists a profile of this type which conserve mass until extinction time, connecting P_1 and P_2.

Chapter 3

Asymptotic behavior for the p-Laplacian equation in domains with holes

3.1 Introduction and description of results

In this chapter we study the asymptotic behavior of the solutions of the parabolic p-Laplacian equation in an exterior domain. More precisely, let $G \subset \mathbb{R}^n$ be a bounded open set with smooth boundary (of class $C^{2,\alpha}$) and let $\Omega = \mathbb{R}^n \setminus G$. We think of G as the "holes". We assume moreover that Ω is connected, which implies no essential loss of generality. We consider the following problem:

$$\left\{ \begin{array}{l} u_t = \Delta_p u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{array} \right. \tag{3.1}$$

where $p > 2$. On the initial data we make the assumptions that $u_0 \in L^1(\Omega)$ and it is nonnegative in Ω. For most of the chapter we also assume that u_0 has compact support in $\overline{\Omega}$.

We are interested in describing the influence of the holes of the domain on the large time behavior of the solutions. For a complete study of the asymptotic behavior in an exterior domain, one has to perform two different steps in the analysis, typical of the technique of matched asymptotics. First, the *outer analysis* gives the asymptotic rates and profiles of the solutions in the far field near infinity. Afterwards, one has to perform the *inner analysis* of the problem, which means studying what happens in the region near the holes (more precisely in bounded subdomains).

We divide the present chapter into three parts as it follows: in a first, shorter part, we deal with the problem posed in dimension $n > p$, where things are easier and there are only technical differences with respect to the similar problem for the PME, treated completely in [37]. Then, a second part will be dedicated to the critical case $n = p$, where the results and the techniques used in the proofs are more involved and mathematically more interesting. Finally, in a third, longer part, we deal with the low dimension case, $n < p$, where things are even more complicated: the asymptotic behavior is given by an anomalous self-similar solution, without explicit formula. In this case, the asymptotic mass is zero and the renormalized asymptotic profiles correspond to what is known as self-similarity of the second kind, or self-similarity with anomalous exponents (see also [12]). Let us mention that such novel features do not appear in the study of large-time behavior of the solutions of the porous medium equation recently done in [37] and [69], though a number of other properties are common. The proof of uniqueness of the rescaled asymptotic profile needs an involved topological argument.

No such division into ranges occurs for the p-Laplacian equation posed in the whole space or in a bounded domain, beyond the basic requirement that $p > 2$ that implies finite propagation speed. Therefore, the present division reflects the varying influence of the holes, that becomes more dramatic when the dimension decreases, since singularities in the limit become stronger. The material of this chapter has been published into two papers, [80] for the cases $n \geq p$ and [81] for $n < p$.

Preliminaries. In order to describe the asymptotic behavior, we need to introduce some preliminary facts and results concerning the parabolic p-Laplacian equation. A very important class of solutions of the p-Laplacian equation consists of the so-called source-type solutions, which are very similar to the ZKB solutions of the porous medium equation. They have the form

$$B_C(x,t) = t^{-\alpha} F_C(y), \tag{3.2}$$

where $y = x\,t^{-\beta}$ and

$$F_C(y) = \left(C - k|y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}. \tag{3.3}$$

The function F_C is called the profile of the source-type solution, and the exponents α and β are the self-similarity exponents. In our case they have the values

$$\alpha = \frac{n}{n(p-2)+p}, \qquad \beta = \frac{1}{n(p-2)+p} \tag{3.4}$$

and the parameter k is also known, $k = ((p-2)/p)\beta^{\frac{1}{p-1}}$. The constant C is a free parameter, that gives the height of the solution. We thus have a whole one-parameter family of solutions of the same type. We remark that the source-type solutions we have introduced satisfy the conservation law

$$\int_{\mathbb{R}^n} B_C(x,t)\,dx = \text{constant}$$

for all times. For convenience we call this integral the total mass of the solution, M_C. This is justified when we think of the equation as nonlinear diffusion of a substance with density u, as explained in the Introduction. It is also easy to see that the source-type solutions have as initial trace a Dirac mass, which is $M_C\delta(x)$. Moreover, the connection between the free parameter C and the mass M_C is given by

$$M_C = dC^{\gamma}, \tag{3.5}$$

where $\gamma = \frac{(p-1)n}{p(p-2)\alpha}$ and

$$d = n\omega_n \int_0^\infty (1 - ky^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}} y^{n-1} dy.$$

For more details about the properties of the self-similar solutions, that is,

$$U(x,t) = t^{-\alpha} F(\xi), \quad \xi = xt^{-\beta} \tag{3.6}$$

having a compactly supported profile F, the reader may study the previous Chapter 2 of the present memoir or [78].

Starting from an arbitrary solution u of the p-Laplacian equation, we consider the family of scaled versions of u:

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t). \tag{3.7}$$

Then, by a straightforward calculation, one can check that starting from a solution u of the p-Laplacian equation, we produce an entire family of solutions of the same equation that are zoomed versions of the initial one. We remark that all the self-similar solutions of the form (3.6) enjoy the nice property of being invariant to the scaling above.

We now introduce the weak formulation of the p-Laplace equation. Let $Q_T = \Omega \times (0,T]$.

Definition 3.1. *A function $u \in C((0,T] : W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ is a weak solution of problem* (3.1) *on [0,T] if for any test function $\Phi \in C^{2,1}(\overline{Q_T})$ with compact support in $\overline{Q_T}$ and $\Phi = 0$ on $\partial\Omega \times (0,T]$, it satisfies the integral identity*

$$\begin{aligned}\int_\Omega u(x,t)\Phi(x,t)dx = &\int_0^t \int_\Omega (u(x,s)\Phi_s(x,s) - |\nabla u|^{p-2}\nabla u(x,s) \cdot \nabla\Phi(x,s))dxds \\ &+ \int_\Omega u_0(x)\Phi(x,0)dx\end{aligned} \tag{3.8}$$

for any $t \in [0,T]$. We say that u is a weak solution of (3.1) *on $[0,\infty)$ if there is a weak solution in the sense above on $[0,T]$ for any $T > 0$.*

The definition of weak sub- and supersolution follows as usual, by replacing in Definition 3.1 the equality by the corresponding inequalities $\leq$ or $\geq$ and considering only nonnegative test functions. We will also introduce the local weak solutions, i.e weak solutions referred only to the equation, without considering the initial and boundary condition.

Definition 3.2. *A function $u \in C((0,T] : W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ is a local weak solution of problem* (3.1) *on [0,T] if for any test function $\Phi \in C^{2,1}(Q_T)$ with compact support in Q_T, it satisfies the integral identity*

$$\int_0^T \int_\Omega (u(x,t)\Phi_t(x,t) - |\nabla u|^{p-2}\nabla u(x,t) \cdot \nabla\Phi(x,t))dxdt = 0. \tag{3.9}$$

The existence and uniqueness of solutions of the p-Laplacian equation has been widely investigated; a good reference is the book [52], where also the optimal regularity is studied. It can be showed that the nonnegative bounded weak solutions of the p-Laplacian equation are such that u, $|\nabla u| \in C^{\alpha}(Q^{'})$ for some $\alpha > 0$ and $Q^{'} \subset Q$, where $Q = \Omega \times (0, \infty)$.

We will often use in the text the notation $u(t)$ for the function $u(t)(x) = u(x,t)$. We will denote by P_u the positivity set of u and by $\Gamma(t) = \partial P_u(t) \setminus \partial\Omega$ the free boundary of u at time t.

The dipole solution. In the study of the asymptotic behavior to be performed in this chapter we will need another self-similar solution of the p-Laplacian evolution equation that was introduced in the paper [78] with the name of *dipole solution*. Precisely, it is obtained implicitly, through the exact relation between the profiles of the self-similar solutions of the porous medium and the p-Laplace equation. The name dipole comes from the fact that it corresponds in the Hulshof-type sequence of the PLE to the well-known dipole solution of the PME, as explained in Chapter 2; in particular, the mass $M(t) = \int u(x,t)\,dx$, as $t \to 0$ of the solution becomes infinite but at any positive moment is finite.

The profile of the dipole solution is not explicit, as it happens for the Barenblatt solution, and we have to work using only its properties deduced from the analysis made in [78]. In the next lines we will state the properties we need in the sequel. We will denote a particular dipole solution by D. Using the notations in [78], we write

$$D(x,t) = t^{-\alpha_2} F(xt^{-\beta_2}), \tag{3.10}$$

where the self-similarity exponents satisfy the relation:

$$(p-2)\alpha_2 + p\beta_2 = 1, \quad \alpha_2 > 0, \quad \beta_2 > 0, \tag{3.11}$$

but we do not have explicit expressions for them, as in the porous medium case. Actually, such exponents are called *anomalous*, since they are not obtained from some conservation law but as the existence of a special orbit of an associated ODE system, cf. [12]. More precisely, in our case, since we are looking for self-similar solutions of the general form $t^{-\alpha} f(\eta)$, $\eta = |x|t^{-\beta}$, the ODE satisfied by the profile f is

$$\eta^{1-n}(\eta^{n-1}|f'|^{p-2}f')' + \alpha f + \beta\eta f' = 0, \tag{3.12}$$

which can be transformed into an autonomous dynamical system, see Section 2.4 of the present work or Section 4 of [78].

We will also denote by $k_2 = \alpha_2/\beta_2$ the associated "eigenvalue". From [78], we deduce that to this eigenvalue corresponds a whole orbit of solutions of dipole-type, and, moreover, all their profiles are obtained from a particular representative F through a simple rescaling:

$$F_{\lambda}(\eta) = \lambda^{p} F(\lambda^{2-p}\eta), \quad \forall\, \lambda > 0, \tag{3.13}$$

hence we will denote in the sequel the members of this orbit by F_{λ} (the profile), and D_{λ} (the solution corresponding to the profile F_{λ}). We remark that the scaling is monotone in λ, in the sense that if $\lambda_1 < \lambda_2$, then both the support and the height of D_{λ_1} are less than those of D_{λ_2}. When the index λ is missing, we will understand $\lambda = 1$. Using Theorem 2.1 from Chapter 2 and the behavior near $x = 0$ of the corresponding solutions of the porous medium equation (see Chapter 2 or the papers [24], [75], [78] where the calculations are given), we also obtain that $F_\lambda(0) = 0$, but its derivative is singular at $\eta = 0$. More precisely, near $\eta = 0$ we have $F'(\eta) \sim \eta^{-(n-1)/(p-1)}$, hence

$$F(\eta) \sim \eta^{(p-n)/(p-1)}, \quad \text{as} \quad \eta \sim 0, \tag{3.14}$$

and we will denote by C_λ the limit

$$C_\lambda = \lim_{\eta\to 0} \frac{F_\lambda(\eta)}{\eta^{(p-n)/(p-1)}} \tag{3.15}$$

Moreover, the dipole profile exists in the sense of weak solution in the whole space only in dimension $n < p$; for $n > p$ the profile develops a singularity at $\eta = 0$, and for $n = p$ it coincides with the Barenblatt solution. We illustrate this bifurcation in Figure 3.1 below, where the dashed line represents the dipole exponents.

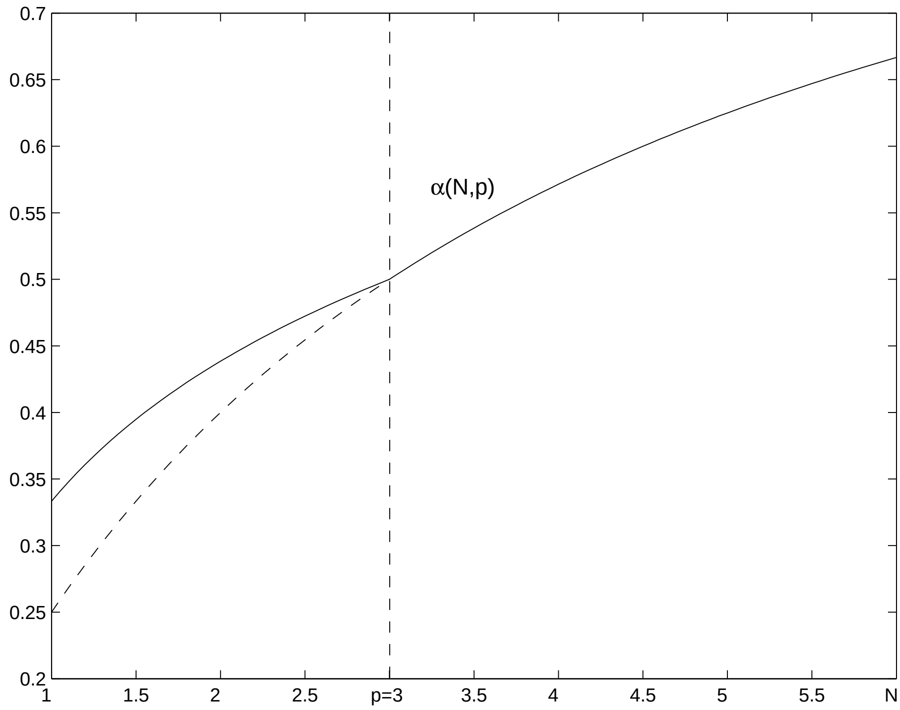

Figure 3.1: Bifurcation of exponents at $N = p$. Experiment for $p = 3$.

Even in dimension $n = 1$ the dipole solution is not explicit. Actually, in that case, a simple differentiation leads to a self-similar solution of the Porous Medium Equation with compactly supported profile and lap number 2. Bernis, Hulshof and Vázquez have studied that solution in [24] and shown that the similarity exponents are *anomalous* in the sense described above.

Outline of results. We will describe in few words the main results of this chapter, in the three different cases $n > p$, $n = p$ and $n < p$.

Case $n > p$. We prove by a scaling argument that the outer analysis is given by the profile of a particular source-type solution, of the form (3.6) and with the exponents given by (3.4). We calculate the constant C_0 that identifies the profile inside the family F_C and prove that the rescaled function $v(x,t) = t^\alpha u(x,t)$ converges to F_{C_0} uniformly in outer sets of the form $|x| \geq \delta$. We point out that there seems to be no conservation law from which the asymptotic constant C_0 may be derived a priori. The study is performed in Section 3.2 and the main result is Theorem 3.1.

Case $n = p$. The analysis of this borderline case is more involved, and we use further techniques from dynamical systems. This makes it mathematically more interesting. The asymptotic profile will be similar to the one of a source-type solution, but we have to introduce a logarithmic correction in order to insure that the total mass disappears in the end. We will get a profile of the form

$$U(x,t) = t^{-\alpha} \left(C(t) - k \left(\frac{|x|}{t^\beta} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}, \tag{3.16}$$

where the dependence of the "free parameter" in time is given by

$$C(t) = C_0 (\log t)^{-\frac{p-2}{(p-1)^2}}. \tag{3.17}$$

Here the self-similarity exponents become

$$\alpha = \frac{1}{p-1}, \quad \beta = \frac{1}{p(p-1)}.$$

We deduce that in the critical case, the solution decays in time like $C_1 (t \log t)^{-1/(p-1)}$ and its support expands like $|x| \sim C_2 t^\beta (\log t)^{-(p-2)/p(p-1)}$. Using (3.5), this gives a mass variable in time with the law $M(t) = C/\log(t)$. We prove that the outer asymptotic behavior of general solutions is given by a profile of this type. The study is performed in Section 3.3 and the main result is Theorem 3.2.

Case $n < p$. We begin by constructing the sub- and supersolutions that we will use later to obtain optimal barriers. Then, we prove that the outer limit is given by a particular dipole profile by identifying the precise scaling factor λ in (3.13). The convergence to the outer profile is uniform in all sets of the form $\{|x| \geq \delta t^\beta\}$ for any $\delta > 0$ sufficiently small. The main result is stated as Theorem 3.3, and the proof takes up Subsections 3.4.2, 3.4.4 and 3.4.8. We will use a different technique, based on the construction of optimal barriers and delicate comparisons. An important step in the analysis is the proof of lack of contact between special solutions, that relies on a delicate use of Harnack principle for degenerate parabolic equations with variable coefficients in space and time, that is due to [43]. The end steps rely on accurate tail analysis. The whole process of proving uniqueness is much more difficult than corresponding similar problems like in [37] where a conservation law is available to determine the asymptotic parameter. Here this law is replaced by a delicate topological study. The outer analysis ends with the convergence of the supports and interfaces, which follows as a consequence of the general uniqueness proof.

Precedents. A complete study of the asymptotic behavior of the p-Laplacian equation posed in the whole space was done by Kamin and Vázquez in [89]. There are a number of works on the problem of evolution in a domain with holes when the equation is the linear heat equation or the porous medium equation. In the case of the linear heat equation the analysis is made easier by the possibility of using integral representation of the solutions, cf. Ishige [83] and [84]. In the case of the porous medium equation, the asymptotic behavior in the whole space is well known cf. [131], while the asymptotic behavior for the Dirichlet problem with zero boundary condition in domains with holes was treated by Brändle at al. [37] and by Gilding and Goncerzewicz in [70] and [69]. In comparison with these works, the absence of a conservation law makes the asymptotic analysis in the p-Laplacian case more involved. On the other hand, Quirós and Vazquez [112] had treated the case of non-homogeneous boundary conditions and showed that the asymptotical results are quite different.

For the low dimension case of the PME, it was proved in [69] that in dimension $n < 2$ there is a big difference in the asymptotic behavior with respect to the case $n \geq 2$. We point out that, since in the porous medium case the only subcritical dimension is $n = 1$, the analysis in this case is similar to studying the porous medium equation on a half-line. On the contrary, in the case of the p-Laplacian evolution equation with p large, there can be many space dimensions in the range $1 \leq n < p$, making the analysis more interesting for the applications.

3.2 Case of large dimensions, $n > p$

The analysis follows the outline of the proof of paper [37] for the porous medium equation, hence we will be rather sketchy.

3.2.1 Sub- and supersolutions. Outer analysis

In this subsection we describe some appropriate sub- and supersolution that will have the same decay in time as the general solution. The construction is based on the source-type solutions presented above, but with some necessary changes.

Supersolutions. As supersolutions, we will consider the Barenblatt functions B_C already defined, with a certain delay in time, $U_{C,\tau}(x,t) = B_C(x, t+\tau)$, $\tau > 0$. It is well known that they are weak solutions of the p-Laplace equation and they become supersolutions for the problem (3.1), since they are positive on the boundary of the hole ∂G. Moreover, by well-known comparison arguments, for any compactly supported solution u of the p-Laplacian equation, there exist constants $C, \tau > 0$ such that $u(x,t) \leq B_C(x, t+\tau)$ at any time. We recall that the parameter $C > 0$ is related to the constant mass M_C of the Barenblatt function by

$$C = c(p,n) M^{\frac{p(p-2)}{p-1}\beta}.$$

Subsolutions. Defining subsolutions is more involved, and we will follow a general idea of construction that has been used in the paper [37] for the porous medium equation. We remark that the Barenblatt functions, although they have a good behavior at infinity, can not be used as subsolutions of the boundary value problem, since they are positive on ∂G. The idea we follow is to consider another local subsolution, which is good near the hole, and then to combine them. A good starting point is to consider a function with separated variables, whose x-part is the fundamental solution of the p-Laplace operator, and the part in t has the expected decay. We define:

$$U(x,t) = Ct^{-\alpha}\left(1-\left(\frac{R}{|x|}\right)^{\frac{n-p}{p-1}}\right)_+ . \tag{3.18}$$

By choosing R such that $G \subset B(0,R)$, we get the desired behavior of H near ∂G. To combine these functions, we assume a delay in time $\tau > 0$ in order to avoid problems at $t = 0$ and we change H in order to be dominated by the Barenblatt function far from the hole. We set:

$$U_\tau(x,t) = C(t)(t+\tau)^{-\alpha}\left(1-\left(\frac{R}{|x|}\right)^{\frac{n-p}{p-1}} - a\frac{(|x|-r)_+^4}{(t+\tau)^l}\right)_+ , \tag{3.19}$$

$$B_{C_0,\tau}(x,t) = (t+\tau)^{-\alpha}\left(C_0 - k\left(\frac{|x|}{(t+\tau)^\beta}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} , \tag{3.20}$$

where R, r, a, C_0 and l are positive parameters, which are free for the moment. We observe that both subsolutions have free boundaries and we denote by $R_1(t)$ and $R_2(t)$ the radii of their free boundaries. We choose $C(t) = K(1+(t+\tau)^{-\sigma})$, where $\sigma > 0$. We remark that

$$\max B_{C_0,\tau} = C_0^{(p-1)/(p-2)}(t+\tau)^{-\alpha}, \quad \forall t > 0,$$

and obviously

$$\max U_\tau \geq K(1+(t+\tau)^{-\sigma})(t+\tau)^{-\alpha}\left(1-\left(\frac{R}{r}\right)^{(n-p)/(p-1)}\right).$$

We choose $r > R$ and we can insure that $\max B_{C_0,\tau} \leq \max U_\tau$ at $|x| = r$ by choosing K sufficiently large. In this way the two subsolutions will intersect each other in a point $r^*(t)$ depending on time and after that intersection we insure that the Barenblatt subsolution dominates and $r^*(t) \leq R_1(t)$. Now we can finally define our family of subsolutions:

$$V_{C_0,\tau}(x,t) = \begin{cases} 0, & \text{if } |x| < R \text{ or } |x| > R_2(t), \\ U_\tau(x,t) & \text{if } R \leq |x| \leq r^*(t), \\ B_{C_0,\tau} & \text{if } r^*(t) \leq |x| \leq R_2(t). \end{cases} \tag{3.21}$$

It is easy to check that $V_{C_0,\tau}$ is a subsolution for sufficiently large times $t > t_0 > 0$, provided $0 < \sigma < l-1$. The next technical result, whose proof follows exactly the same lines as the proof of Lemma 3.1 in [37], shows that this rather complicated construction is good for our purposes.

Proposition 3.1. *For any solution $u(x,t)$ of (1.1), there exists a choice of the parameters C_0, τ, a, R, r and a time $t_0 > 0$ such that for any time $t > t_0$ and $x \in \Omega$ we have $V_{C_0,\tau}(x,t) \leq u(x,t)$.*

With these constructions, we can pass to the study of the outer analysis.

Theorem 3.1. *For $n > p$, if u is a weak solution of the problem (3.1), there exists a constant $C_0 > 0$ such that*

$$\lim_{t\to\infty} t^{\alpha}|u(x,t) - B_{C_0}(x,t)| = 0 \tag{3.22}$$

uniformly far from the hole, i.e. on sets of the form $\{|x| \geq \delta t^{\beta}\}$, where $\delta > 0$ is sufficiently small.

Proof. We follow the general program proposed by one of the authors in [128] for studying the asymptotic behavior of the nonlinear diffusion problems. This program has four different steps: in the first step we consider the family of rescaled solutions u_λ and we obtain compactness estimates for it. As a consequence, there exists a limit point u_∞ of u_λ. In the second step we prove that any limit point is a Barenblatt function. In the third step we prove that the convergence along subsequences is uniform on compact sets. The proof of the first three steps is very similar to that of Theorem 3.1 in [37], but we describe it in detail for reader's convenience.

1. Scaling and compactness. We define the family of rescaled solutions

$$u_\lambda(x,t) = \lambda^{\alpha}u(\lambda^{\beta}x, \lambda t),$$

with α, β the Barenblatt exponents for the PLE, which leaves invariant the Barenblatt functions. We thus construct a family u_λ of solutions that is uniformly bounded by some Barenblatt solution, as shown above. By standard compactness of the PLE, cf. [52], there exists a subsequence λ_k and some function u_∞ such that $u_{\lambda_k} \to u_\infty$ uniformly on compact sets in $\mathbb{R}^n \setminus \{0\} \times (0,\infty)$. It follows also from this and the definitions that u_∞ is a weak solution of the PLE in $\mathbb{R}^n \setminus \{0\} \times (0,\infty)$, but we still have no information about its behavior at the origin.

2. The limit is a Barenblatt solution. This is the more important step. From Proposition 3.1, we know that there exist parameters C_0 and $C > 0$ such that

$$V_{C_0}(x,t) \leq u(x,t) \leq B_C(x,t),$$

for sufficiently large times, with V_{C_0} given by Proposition 3.1. Rescaling this expression and passing to the limit as $\kappa \to \infty$ on the subsequence λ_k in Step 1, we obtain that

$$B_{C_0}(x,t) \leq u_\infty(x,t) \leq B_C(x,t). \tag{3.23}$$

From this, we first derive that u_∞ is a nontrivial solution of the PLE in $\mathbb{R}^n \setminus \{0\}$ and it is bounded for all positive times. From this, we next show that in fact the possible singularity at $x = 0$ is removable and u_∞ can be extended as a local weak solution in the whole $\mathbb{R}^n$ for positive times.

Take a cutoff function ψ with $0 \leq \psi \leq 1$, such that ψ vanishes near $x = 0$ and $\psi \equiv 1$ for $|x| \geq 1$. Let $\psi_r(x) = \psi(x/r)$. Let $\xi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and plug the test function $\Phi(x,t) = \xi(x,t)\psi_r(x)$ into the weak formulation of the PLE. Notice that Φ is admissible as a test function in this formulation with respect to the local weak solution u_∞, since it vanishes near the origin. We obtain:

$$\int_\Omega u_\infty(x,t)\xi(x,t)\psi(x/r)\,\mathrm{d}x - \int_\Omega u_\infty(x,0)\xi(x,0)\psi(x/r)\,\mathrm{d}x$$
$$= \int_0^t \int_\Omega \big(u(x,s)\xi_s(x,s)\psi(x/r) - |\nabla u|^{p-2}\nabla u(x,s)\cdot\nabla\Phi(x,s)\big)\,\mathrm{d}x\,\mathrm{d}s.$$

We remark that, from the left-hand side, the second term vanishes, and the first one tends to 0 as $r \to 0$ (after the obvious change of variable $y = x/r$). On the other hand, to estimate the second, we observe that

$$\int_0^t \int_\Omega |\nabla u|^{p-2}\nabla u(x,s)\cdot\nabla\Phi(x,s)\,\mathrm{d}x\,\mathrm{d}s \leq \left(\int_0^t \int_\Omega |\nabla u|^p\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p-1}{p}} \left(\int_0^t \int_\Omega |\nabla\Phi(x,s)|^p\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{p}}$$
$$\leq r^{\frac{n-p}{p}}\|\nabla u\|_p^{p-1}\left(\int_0^t \int_\Omega |\nabla\Phi(y,s)|^p\,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{1}{p}},$$

which tends to 0 as $r \to 0$, precisely because we assume $n > p$. This means that the singularity at $x = 0$ is removable and u_∞ is a local weak solution in the whole $\mathbb{R}^n$ for positive times.

On the other hand, by (3.23), we obtain that

$$\mathrm{supp}B_{C_0}(t) \subseteq \mathrm{supp}u_\infty(t) \subseteq \mathrm{supp}B_C(t),$$

for any $t > 0$. Passing to the limit as $t \to 0$ in the sense of initial trace, it follows easily that the trace should be a finite measure supported only at the origin, that is, a multiple of the Dirac delta. By the uniqueness theorem of Kamin and Vázquez [89], u_∞ must be a Barenblatt solution of the PLE, with parameter $C_1 \in (C_0, C)$.

3. Convergence along subsequences. By the invariance of the Barenblatt solutions under scaling, we have

$$|u_{\lambda_k}(y,1) - B_{C_1}(y,1)| = \lambda_k^\alpha |u(x,\lambda_k) - B_{C_1}(x,\lambda_k)|,$$

where $x = \lambda_k^\beta y$. Thus, the uniform convergence of u_{λ_k} to $u_\infty = B_{C_1}$ in sets of the form $\{|y| \geq \delta\}$ implies the uniform convergence of $u(x,t_k)$ to $B_{C_1}(x,t_k)$ in sets of the form $\{|x| \geq \delta t_k^\beta\}$, once we let $t_k = \lambda_k$, that is, the result of Theorem 3.1 on a subsequence.

4. Independence of the chosen subsequence. Mass analysis. We still have to prove the independence of the limit w.r.t. the subsequence of times. Since the conservation law used in [37] does not hold in our case, we will prove differently this last step. From the previous steps, we already have that $u(x, t_k) \sim B_{C_0}(x, t_k)$ as $k \to \infty$ on some subsequence of time. Suppose that there exists two subsequences $t_{k,1}$ and $t_{k,2}$ such that

$$\lim_{k\to\infty} t_{k,1}^{\alpha} |u(x, t_{k,1}) - B_{C_1}(x, t_{k,1})| = 0 \tag{3.24}$$

and

$$\lim_{k\to\infty} t_{k,2}^{\alpha} |u(x, t_{k,2}) - B_{C_2}(x, t_{k,2})| = 0 \tag{3.25}$$

uniformly on sets of type $\{x \in \Omega : |x| \geq \delta t_{k,1}^{\beta}\}$, resp. $\{x \in \Omega : |x| \geq \delta t_{k,2}^{\beta}\}$, $\delta > 0$, where C_1, C_2 are positive constants.

Let $M(t) = \int_{\Omega} u(x, t)\mathrm{dx}$ be the mass at time t. It is well-known that, since we have homogeneous Dirichlet boundary conditions, the mass is decreasing in time. Hence, there exists $M = \lim_{t\to\infty} M(t)$. The explicit lower bound (subsolution) implies that this *asymptotic mass* $M > 0$. With this information we can identify the limit. Indeed, we pass to the limit in the relations (3.24) and (3.25), but in the renormalized variable $y = xt^{\beta}$. In this variable, these two relations are written as

$$|u_{t_{k,1}}(y, 1) - B_{C_1}(y, 1)| \to 0, \quad |u_{t_{k,2}}(y, 1) - B_{C_2}(y, 1)| \to 0, \tag{3.26}$$

with pointwise convergence in $\mathbb{R}^n$ and uniform convergence in sets of the form $\{|y| \geq \delta\}$ with $\delta > 0$ small. By integrating in y in (3.26) and using the dominated convergence theorem, we obtain that the mass of the Barenblatt solutions $B_{C_1}(\cdot, 1)$ and $B_{C_2}(\cdot, 1)$ is the same, i.e. $M = M_{C_1} = M_{C_2}$. This implies $C_1 = C_2$, hence we have a unique limit independent on the subsequence. $\square$

We remark that this argument does not allow for quantitative estimates concerning the mass lost in the evolution. We only obtain the correct decay in time and the profile.

3.3 Critical case $n = p$

The case $n = p$ provides an important difference with the previous case in the general theory of the p-Laplacian, since the fundamental solution of the equation is $C|x|^{-(n-p)/(p-1)}$ for $n > p$ and $\log|x|$ for $n = p$. In this way, the dimension $n = p$ corresponds to the case $n = 2$ for the usual Laplacian. On the other hand, the hole starts to play an important role. Indeed, by performing the rescaling as before, we arrive to a solution with a singularity at $x = 0$, but this singularity is no more removable. In the proofs, we will suppose that p is an integer and the problem has physical sense. In radial variables any dimension makes sense theoretically, but the proofs are perfectly similar, since all the profiles that we use for comparison are radial.

3.3.1 Formal derivation of the logarithmic correction

In this part we follow an idea of [69] based on some formal calculations using a weighted integral. The rigourous proof will be different, but this calculation helps us to conjecture the correct asymptotic profile. For this calculation, we need to pass to the radial variables and consider (in any dimension $n \geq p$) the same problem with radially symmetric initial data $u_0(r)$, $r = |x|$, where u_0 is compactly supported and bounded. Similarly to the usual convolution with the Green kernel, we define the following weighted integral for the Barenblatt solutions:

$$Z : [1, \infty) \times (0, \infty) \to \mathbb{R}, \quad Z(r,t) = \int_r^\infty k(x,r) B_C(x,t)\, dx \tag{3.27}$$

where the kernel k is given by the fundamental solution:

$$k(x,r) = \begin{cases} x^{p-1} r^{p-n} (x^{n-p} - r^{n-p})/(n-p), & \text{if } \ n > p, \\ \\ x \log(x/r), & \text{if } \ n = p. \end{cases} \tag{3.28}$$

Our goal in this subsection is to calculate the behavior when $t \to \infty$ of $Z(r,t)$ and to remark what are the differences that appear when passing from $n > p$ to $n = p$. We first calculate it for $n > p$. We have:

$$\begin{aligned} Z(r,t) &= \frac{1}{n-p} \int_r^\infty x^{p-1} r^{p-n} (x^{n-p} - r^{n-p}) t^{-\alpha} (C - k(x/t^\beta)^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}\, dx \\ &= \frac{1}{n-p} \int_{r/t^\beta}^\infty (y^{n-1} r^{p-n} - y^{p-1} t^{p\beta - \alpha}) (C - k y^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}\, dy. \end{aligned}$$

Since $\alpha = n\beta$ and $n > p$, it follows that $p\beta - \alpha < 0$, hence

$$\lim_{t \to \infty} Z(r,t) = \frac{r^{p-n}}{n-p} \int_0^\infty y^{n-1} (C - k y^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}\, dy < \infty.$$

This follows from the fact that the asymptotic profile is the Barenblatt and the weighted integral should have a finite limit as $t \to \infty$, obtained by canceling the time from the independent integral in y. This also should pass in the case $n = p$ if we want to obtain the correct profile. Let us pass to the case $N = p$ and calculate the same integral:

$$
\begin{aligned}
Z(r,t) &= \int_r^\infty x^{p-1} \log \frac{x}{r} t^{-\alpha} (C - k(x/t^\beta)^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}} dx \\
&= \int_{r/t^\beta}^\infty y^{p-1} t^{(p-1)\beta} t^{-\alpha} \log \frac{yt^\beta}{r} (C - ky^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}} t^\beta dy \\
&= \int_{r/t^\beta}^\infty y^{p-1} \log \frac{yt^\beta}{r} (C - ky^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}} dy.
\end{aligned}
$$

We remark that $\lim\limits_{t\to\infty} Z(r,t) = \infty$, for any $r > 1$, with logarithmic rate. For the divergence we introduce in the calculation a correction of logarithmic type, in order to compensate and obtain a finite limit. It is convenient to insert this correction into the form of the Barenblatt solution. Let us consider

$$
\overline{B}_C(x,t) = t^{-\alpha} \left(C(\log t)^\gamma - k \left(\frac{x}{t^\beta} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}.
$$

Analyzing the previous calculation of $Z(r,t)$, in order to compensate, we need to set $\gamma = -\frac{p-2}{(p-1)^2}$. In order to avoid the problems with the singularity of the logarithmic part, we will permit a delay in time $T > 0$. Hence, we conjecture that the outer asymptotic behavior of solutions in the case $n = p$ is given by a function from the family:

$$
\begin{aligned}
U_T(x,t;C) &= (t+T)^{-\alpha} \left(C(\log(t+T))^{-\frac{p-2}{(p-1)^2}} - k \big(\frac{|x|}{(t+T)^\beta} \big)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}} \\
&= [(t+T)\log(t+T)]^{-\frac{1}{p-1}} \left(C - k \big(\frac{|x|}{(t+T)^\beta} \big)^{\frac{p}{p-1}} \log(t+T)^{\frac{p-2}{(p-1)^2}} \right)_+^{\frac{p-1}{p-2}},
\end{aligned}
\tag{3.29}
$$

where in this case

$$
\alpha = \frac{1}{p-1}, \quad \beta = \frac{1}{p(p-1)}, \quad k = \frac{p-2}{p} \left(\frac{1}{p(p-1)} \right)^{\frac{1}{p-1}},
$$

see [127]. In the following sections we will prove this claim.

A convenient way of visualizing the corrected asymptotic behavior is in terms of the mass. In our case

$$
M(t) = \int_\Omega U_T(x,t;C) dx = \frac{C}{\log t}. \tag{3.30}
$$

In fact, we see that the logarithmic correction we introduce is exactly the inverse of the number γ which connects the "free parameter" of a general Barenblatt solution and its mass, see (3.5). A similar expression of mass decay happens in other critical cases, see [63] and [69]. Contrary to the latter case, here we do not have an exact conservation law to obtain the precise decay and the constant. Hence, we have to use a different technique.

3.3.2 Subsolutions

The construction of subsolutions of (3.1) is technical and follows the same idea as in Section 3.2. Our profiles U_T are indeed subsolutions of the equation, but not for the boundary-value problem (3.1), since they do not vanish on the boundary of the hole. We have to combine these profiles with other profiles that are compactly supported, vanish on the boundary of Ω and dominate near the hole. We consider the family of profiles

$$H_T(x,t) = A(t+T)((T+t)\log(T+t))^{-\frac{1}{p-1}} \left(\log(|x|-r_0) - \frac{a(|x|-r_1)_+}{(T+t)^l}\right)_+, \qquad (3.31)$$

where $A(t+T) = K(1+(t+T)^{-\sigma})$ and the parameters a, r_0, r_1, K, l and σ are free to choose.

The idea is to intersect U_T and H_T with the correct angle, i. e. such that the profile U_T dominates far from the hole. We ask that $\max U_T \leq \max H_T$. But

$$\max U_T \leq ((T+t)\log(T+t))^{-\frac{1}{p-1}} C^{\frac{p-1}{p-2}}$$

and

$$\max H_T \geq H_T(r_1) = ((T+t)\log(T+t))^{-\frac{1}{p-1}} K(1+(t+T)^{-\sigma})\log(r_1-r_0),$$

hence it is enough to choose K such that $K\log(r_1-r_0) = 2C^{(p-1)/(p-2)}$ and $r_1 > r_0$.

Denote by $R_1(t)$ the radius of the interface of H_T and by $R_2(t)$ the radius of the interface of U_T. Then $R_1(t)$ is the unique solution of the equation

$$a(r-r_1) = (T+t)^l \log(r-r_0) \qquad (3.32)$$

with $r > r_1$ and

$$R_2(t) = \left(\frac{C}{k}\right)^{\frac{p-1}{p}} (t+T)^{\beta} \log(t+T)^{-\frac{p-2}{p(p-1)}}. \qquad (3.33)$$

We choose C and T such that $R_2(t) > R_1(t)$, for all $t > 0$. Then there exists $r^*(t)$ such that $1 < r^*(t) < R_1(t) < R_2(t)$, for all $t > 0$, such that H_T and U_T intersect at a distance $r^*(t)$. We define

$$V_T(x,t;C) = \begin{cases} 0, & \text{if } |x| < 1+r_0 \text{ or } |x| > R_2(t), \\ H_T(x,t), & \text{if } 1+r_0 \leq |x| \leq r^*(t), \\ U_T(x,t;C), & \text{if } r^*(t) \leq |x| \leq R_2(t). \end{cases} \qquad (3.34)$$

It follows by direct calculation, taking into account that $\Delta_p \log|x| = 0$, that $V_T(x,t;C)$ is indeed a subsolution. The next technical result shows that these subsolutions are good.

Proposition 3.2. *For any solution u of (3.1), there exists a time $t_0 > 0$ large and a choice of the parameters C, T, a, r_0, r_1, l such that $V_T(x,t;C) \leq u(x,t)$, for all $x \in \Omega$ and $t \geq t_0$.*

Proof. **Step 1:** Show that there exists a time $t_0 > 0$ and a choice of the parameters such that $V_T(x,t_0;C) \leq u(x,t_0)$, $\forall x \in \Omega$.

Take $t_0 > 0$ such that $\mathrm{Int}(\mathrm{supp}\, u(\cdot,t_0))$ is large enough (by well-known results, it enlarges as time passes). Choose r_0, r_1 such that the annulus $\overline{W_{r_0,r_1}(0)} \subset \mathrm{Int}(\mathrm{supp}\, u(\cdot,t_0))$. Then choose the constant C measuring the height of the subsolution V_T such that V_T lies below u at time t_0. To choose the delay T, we ask that $\operatorname{supp} V_T(\cdot, t_0; C) = \overline{W_{r_0+1,R_2(t_0)}(0)} \subset \operatorname{supp} u(\cdot, t_0)$. Choose $R_2(t_0)$ and r_1 such that $r_1 < R_2(t_0) = \xi_+(t_0) - \varepsilon < \xi_+(t_0)$ and $r_1 < \xi_+(t_0) - 2\varepsilon$, where $\xi_+(t_0) = \sup\{r > 0 : B(0,r) \subset \operatorname{supp} u(\cdot, t_0)\}$. From this choice, we find T as the solution of the equation

$$\left(\frac{C}{k}\right)^{\frac{p-1}{p}} (T+t_0)^{\frac{1}{p(p-1)}} \log(T+t_0)^{-\frac{p-2}{p(p-1)}} = \xi_+(t_0) - \varepsilon. \tag{3.35}$$

In order to have a unique solution of (3.35), we have to increase again the time t_0 such that $\log t_0 \geq (p-2)$. Then the function $h(T) = (T+t_0)\log(T+t_0)^{2-p}$ is increasing and the uniqueness of the solution is obvious. The choice of a comes from the condition $R_1(t_0) < R_2(t_0)$.

Step 2: For any $t \geq t_0$, $V_T(x,t;C) \leq u(x,t)$ for all $x \in \Omega$. To do this, we use well-known arguments of comparison, starting from $t = t_0$ as initial time. Since the subsolution and the solution are separate, the only thing that we have to prove is that the above construction can be done, i.e. $R_1(t) < R_2(t)$, for all $t > t_0$. We use the standard procedure: let $g(t) = R_2(t) - R_1(t)$. Then $g(t_0) > 0$ and suppose there exists a first time $t_1 > t_0$ such that $g(t_1) = 0$. Then $R_2(t_1) = R_1(t_1)$ and $g'(t_1) \leq 0$.

On the other hand, $g'(t_1) = R_2'(t_1) - R_1'(t_1)$. By differentiating in (3.33), we obtain

$$R_2'(t_1) = \frac{\beta}{t_1+T}\frac{\log(t_1+T)-(p-2)}{\log(t_1+T)}R_2(t_1).$$

To obtain $R_1'(t_1)$, we differentiate in the equation (3.32) and from a straightforward calculation and taking into account that $R_1(t_1) = R_2(t_1)$, we have:

$$\begin{aligned} g'(t_1) &= \frac{1}{t_1+T}\left(\beta R_2(t_1)\frac{\log(t_1+T)-(p-2)}{\log(t_1+T)} - \frac{la(R_2(t_1)-r_1)(R_2(t_1)-r_0)}{a(R_2(t_1)-r_1)-(t_1+T)^l}\right) \\ &= \frac{1}{(t_1+T)[a(R_2(t_1)-r_1)-(t_1+T)^l]}\left[a\left(\beta\frac{\log(t_1+T)-(p-2)}{\log(t_1+T)} - l\right)R_2(t_1)^2\right. \\ &\left. + \left((t_1+T)^l + la(r_1+r_0) - a\beta r_1\frac{\log(t_1+T)-(p-2)}{\log(t_1+T)}\right)R_2(t_1) - r_1r_0la\right]. \end{aligned}$$

By enlarging the initial time t_0 (hence at the same time t_1) and choosing $l < \beta$, we obtain that $g'(t_1) > 0$, in contradiction with the assumptions on t_1. Hence $R_1(t) < R_2(t)$, for all $t > t_0$. $\square$

3.3.3 Continuous rescaling and supersolutions

In this section we prove that indeed the functions $U_T(x,t;C)$ obtained formally are the correct asymptotic profiles of the general nonnegative solutions of the p-Laplacian equation in dimension $n = p$. The main difficulty is that the profile $U_T(x,t;C)$ is not a self-similar solution of the equation, but a subsolution, hence we can not use the classical comparison techniques that hold only for solutions.

Justified by the previous comments, we replace the comparison technique by the technique of continuous rescaling, see [62] or [127]. We set:

$$\eta = x(t+T)^{-\beta}\log(t+T)^{\frac{p-2}{p(p-1)}}, \ \ \tau = \log(t+T), \ \ v(\eta,\tau) = ((t+T)\log(t+T))^{\frac{1}{p-1}}u(x,t). \tag{3.36}$$

The main difference between this scaling and the one of the first section is that the zoom factor changes continuously with time. This justifies the name of continuous rescaling. The generality of this technique comes from the fact that the zoom factors may be changed from problem to problem and in this way the method is very flexible. Moreover, in general, after a good time-dependent rescaling the resulting equation is simpler than the initial one.

In our case, we obtain the new equation satisfied by v:

$$v_\tau = \Delta_p v + \beta\eta\cdot\nabla v + \alpha v - \frac{p-2}{p(p-1)\tau}\eta\cdot\nabla v + \frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}v, \tag{3.37}$$

that we will call in the sequel as the perturbed equation. We associate its autonomous counterpart, which is:

$$v_\tau = \Delta_p v + \beta\eta\cdot\nabla v + \alpha v, \tag{3.38}$$

and will be called the *limit equation*. By these transformations, the profiles $U_T(x,t;C)$ transform into the family

$$F_C(\eta) = \left(C - k|\eta|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p-2}}, \tag{3.39}$$

which are stationary solutions of the limit equation (3.38).

On the other hand, we need some apriori estimates on the general solutions of the p-Laplacian equation. Since we already have enough subsolutions, we need to construct supersolutions that can be compared with any solution, in order to bound the solutions also from above. The construction of a supersolution is rather technical and is given in the next

Proposition 3.3. *For any $C > 0$ sufficiently large, there exists a choice of the free parameters γ, d, b and $q < 0$ such that the following profile:*

$$\begin{aligned}\overline{U}_T(x,t;C) =& ((T+t)\log(T+t))^{-\frac{1}{p-1}}\left(C - k\left(\frac{|x|}{(T+t)^\beta}\log(t+T)^{\frac{p-2}{p(p-1)}}\right.\right.\\ &\left.\left.+\frac{d}{\log(t+T)^\gamma}\right)^{\frac{p}{p-1}}\left(1+\frac{b}{\log(t+T)^\gamma}\right)^{\frac{pq}{p-1}}\right)_+^{\frac{p-1}{p-2}}\end{aligned} \tag{3.40}$$

is a supersolution for the p-Laplacian equation in Ω.

Proof. The proof consists of a very long calculation. In any case, it seems much easier checking it on the rescaled equation (3.37), since the profile $\overline{U}$ changes into the following simpler form:

$$v_+(\eta,\tau) = \left(C - k\left(|\eta| + \frac{d}{\tau^\gamma}\right)^{\frac{p}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}}\right)_+^{\frac{p-1}{p-2}} \tag{3.41}$$

The supersolution condition becomes

$$
\begin{aligned}
0 \leq & \frac{kp}{p-2}\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{1}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}-1}\frac{1}{\tau^{\gamma+1}}\left(d+b|\eta|+\frac{2bd}{\tau^\gamma}\right)\\
&+C\left[p\beta\left(1+\frac{b}{\tau^\gamma}\right)^{pq}+\beta\left(1+\frac{b}{\tau^\gamma}\right)^{pq}\frac{(p-1)d}{|\eta|\tau^\gamma}-\alpha-\frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}\right]\\
&+\frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}k\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{p}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}}\\
&+pk\beta\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{p}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}}\left(1-\left(1+\frac{b}{\tau^\gamma}\right)^{pq}\right)\\
&-k\beta\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{p}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}+pq}\frac{(p-1)d}{|\eta|\tau^\gamma}\\
&+\left(\beta-\frac{p-2}{p(p-1)\tau}\right)\beta^{\frac{1}{p-1}}\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{1}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}}|\eta|\\
&-\beta^{\frac{p}{p-1}}\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{1}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}+pq}.
\end{aligned}
$$

We will prove this complicated inequality by separating it into two parts:
(a) This is the part with the free parameter C characterizing the profile. The inequality that we prove is:

$$
C\left[p\beta\left(1+\frac{b}{\tau^\gamma}\right)^{pq}+\beta\left(1+\frac{b}{\tau^\gamma}\right)^{pq}\frac{(p-1)d}{|\eta|\tau^\gamma}-\alpha-\frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}\right]\geq 0. \tag{3.42}
$$

The main fact is that all these inequalities make sense in the positive part of the profile (3.41), in the rest being trivial. For that, we have to consider that η is bounded, more precisely

$$
|\eta| \leq \left(\frac{C}{k}\right)^{\frac{p-1}{p}}\left(1+\frac{b}{\tau^\gamma}\right)^{-q},
$$

hence fixing $C > 0$, one can choose any number $\gamma < (p-2)/(p-1)$, $q < 0$ and d sufficiently large in order to hold the inequality from (a) at any time $\tau > \tau_0 > 0$ fixed. This is rather easy to achieve.
(b) The inequality formed with the rest of the terms can be a little simplified and written on the form

$$
\begin{aligned}
0 \leq & \frac{kp}{(p-1)\tau^{1+\gamma}}\left(d+\beta|\eta|+\frac{2bd}{\tau^\gamma}\right)+\frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}k\left(|\eta|+\frac{d}{\tau^\gamma}\right)\left(1+\frac{b}{\tau^\gamma}\right)\\
&-k\beta\left(|\eta|+\frac{d}{\tau^\gamma}\right)\left(1+\frac{b}{\tau^\gamma}\right)^{pq}\frac{(p-1)d}{|\eta|\tau^\gamma}\\
&+pk\beta\left(|\eta|+\frac{d}{\tau^\gamma}\right)\left(1+\frac{b}{\tau^\gamma}\right)\left(1-\left(1+\frac{b}{\tau^\gamma}\right)^{pq}\right)\\
&+\left(\beta-\frac{p-2}{p(p-1)\tau}\right)\beta^{\frac{1}{p-1}}|\eta|\left(1+\frac{b}{\tau^\gamma}\right)-\beta^{\frac{1}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{pq+1}.
\end{aligned}
$$

We compare one by one the terms with plus and the terms with minus in the preceding inequality. We have to make a different analysis depending on the values of $|\eta|$. If $|\eta|$ is very small, then we encounter a difficulty in compensating the term with minus where we divide by $|\eta|$. But in this case we will not separate the two inequalities. We remark that this term comes from the term

$$-\beta\left(C-k\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{p}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}}\right)\left(1+\frac{b}{\tau^\gamma}\right)\frac{(p-1)d}{|\eta|\tau}$$

obtained after calculating $\Delta_p v_+$ and simplifying. Since $|\eta|$ is small (for example $|\eta|<1$), for C sufficiently large we can make a decomposition of this term by dividing C and letting $C/2$ in the expression above and introducing only $C/2$ in the corresponding part of the inequality (3.42). In this way, the term in part (b) is compensated directly, by the remaining term with $C/2$, and in (a) we have only to replace d by $2d$.

If $|\eta|>1$, by comparing one by one the terms in the inequality in part (b), we find that it is enough to choose the parameters b and q such that $|q|p^2b>(p-1)d$, where d is already chosen from (a), and $q<0$. With this, the proposition is proved. □

The usefulness of this construction, which is very general (we show in fact that for any $C>0$ we can construct such a supersolution) is illustrated in the following result:

Proposition 3.4. *For any solution u of the p-Laplacian equation in Ω, there exist a constant $C>0$ and a delay $T>0$ sufficiently large such that*

$$u(x,t)\le U_T(x,t;C) \tag{3.43}$$

in the notations introduced above.

Proof. Let $u_0(x)=u(x,0)$ be the initial value of the solution. There exist a delay $T>0$ and a constant $C>0$ such that the function $\overline{U}_T(x,t;C)$ has the following two properties:

(I) $\operatorname{supp} u_0\subset\operatorname{supp}\overline{U}_T(x,0;C)$;

(II) On $\operatorname{supp} u_0$, we have: $u_0(x)\le\overline{U}_T(x,0;C)$.

We say in this case that u and $\overline{U}_T$ are separated at time $t=0$. Since u is a solution of the equation and $\overline{U}_T$ is a supersolution and they are separated at the initial time, a well-known comparison result says that $u(x,t)\le\overline{U}_T(x,t;C)$ for all $x\in\Omega$ and for any time $t>0$. On the other hand, we have:

$$\begin{aligned}\overline{U}_T(x,t;C)&\le((T+t)\log(T+t))^{-\frac{1}{p-1}}\left(C-k|x|^{\frac{p}{p-1}}(\log(t+T))^{\frac{p-2}{(p-1)^2}}\big(1+\frac{b}{(\log T)^\gamma}\big)^{\frac{pq}{p-1}}\right)_+^{\frac{p-1}{p-2}}\\&\le((T+t)\log(T+t))^{-\frac{1}{p-1}}\left(C_T-k|x|^{\frac{p}{p-1}}(\log(t+T))^{\frac{p-2}{(p-1)^2}}\right)_+^{\frac{p-1}{p-2}}\\&=U_T(x,t;C_T),\end{aligned}$$

where $C_T=C\big(1+\frac{b}{(\log T)^\gamma}\big)^{-pq/(p-1)}$. □

This result, together with the one about subsolutions, shows that the family U_T is sufficient to control all the solutions u.

3.3.4 Outer analysis

It will be given by some profile of the form (3.29), depending on the initial data of the problem. The convergence will be uniform away from the hole and with a specified rate. But let us first state the main result of this subsection.

Theorem 3.2. *Let $u(x,t)$ be the unique weak solution of* (3.1) *with initial data $u_0 \in L^1(\Omega)$, nonnegative and compactly supported, in dimension $n = p$. Then there exist a constant C_0 depending on u_0 such that*

$$\lim_{t\to\infty} (t \log t)^{\frac{1}{p-1}} |u(t) - U(\cdot, t; C)| = 0, \tag{3.44}$$

with uniform convergence in any set of the form $\{|x| \geq \delta\lambda(t)\}$, where $\delta > 0$ is sufficiently small and

$$\lambda(t) = t^{\beta}(\log t)^{-\frac{p-2}{p(p-1)}}, \quad \beta = \frac{1}{p(p-1)}.$$

Here $U(\cdot, t; C) = U_1(\cdot, t; C)$.

Proof. The proof consists in an application of the S-theorem (see [62] or [63]). We check below its hypothesis.

(H1). Boundedness and compactness. We need some uniform boundedness and compactness estimates for the orbits $(v(\tau))_{\tau\in\mathbb{R}}$. To obtain that, we use Proposition 3.4, which implies easily that

$$v(\tau) \leq \left(C - k|\eta|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}. \tag{3.45}$$

From (3.45) and the fact that the profiles $(C - k|\eta|^{p/(p-1)})_+^{(p-1)/(p-2)}$ are stationary and bounded uniformly in $L^q(\mathbb{R}^n)$, for all $q \in [1, \infty]$, we deduce similar uniform boundedness estimates for the orbits $v(\tau)$. The compactness estimates follow from the standard results in [52].

(H2). Convergence. Assume that $v(\tau+s) \to w(\tau)$ as $s \to \infty$, and we want to verify which is the equation satisfied by w. We multiply equation (3.37) by any test function $\Phi \in C_0^\infty(\Omega)$ and we integrate in space and in time in $(\tau+s_1, \tau+s_2)$, where $s_2 = s_1 + T$, $T > 0$ fixed. We write the weak formulation of the equation (3.37):

$$\int\limits_{\overline{\Omega}(\tau+s_2)} (v(\tau+s_2) - v(\tau+s_1))\Phi \, \mathrm{d}x = -\int\limits_{\tau+s_1}^{\tau+s_2} \int\limits_{\overline{\Omega}(s)} |\nabla v|^{p-2}\nabla v \cdot \nabla\Phi \, \mathrm{d}x \, \mathrm{d}s$$
$$+ \int\limits_{\tau+s_1}^{\tau+s_2} \int\limits_{\overline{\Omega}(s)} \left[\left(\beta - \frac{p-2}{p(p-1)s}\right)\eta\cdot\nabla v\Phi + \frac{1}{p-1}\left(1 - s^{-\frac{p-2}{p-1}}\right)v\Phi\right] \mathrm{d}x \, \mathrm{d}s.$$

We pass to the limit in this weak formulation with $s_1 \to \infty$ and T fixed. Hence $s_2 \to \infty$ too. By (3.45), the terms in the perturbation go to 0. On the other hand, the left-hand side goes to 0, since from hypothesis, we assume the convergence in time from the beginning at this point. From the right-hand side it remains:

$$T\int_{\mathbb{R}^n} |\nabla v|^{p-2}\nabla v\cdot\nabla\Phi dx+\beta T\int_{\mathbb{R}^n}\eta\cdot\nabla v\Phi dx+\alpha T\int_{\mathbb{R}^n} v\Phi dx=0,$$

which is the weak formulation of the elliptic counterpart of the equation (3.38).

(H3). Reduced uniform stability. This is usually the most difficult hypothesis, and it refers only to the limit equation and to the family of candidate profiles (solutions of it) but in this case this is immediate, since equation (3.38) is also obtained in a standard way from the p-Laplacian equation posed in the whole space (without holes), whose stability property is well known (see [62]).

We introduce the ω-limit of the orbit of v as

$$\omega(v)=\{f\in L^1(\mathbb{R}^n): \text{there exists } \ \tau_j\to\infty, \ \ v(\eta,\tau_j)\to f(\eta)\},$$

where the convergence is taken locally uniformly. By (H1)-(H3), we are in the conditions to apply the S-Theorem and we deduce from it that any $f\in\omega(v)$ is a weak stationary solution of the equation (3.38).

We have now to identify the limit as one of the profiles F_C given by (3.39). For the beginning, we refer to the elliptic equation

$$\Delta_p v+\beta\eta\cdot\nabla v+\alpha v=0 \tag{3.46}$$

satisfied by the elements of $\omega(v)$. It is well-known that the asymptotic profiles of the p-Laplacian equation posed in the whole space are given by the Barenblatt solutions given by (3.2)-(3.3) (see for example [89]). On the other hand, if we use in this case the time-adapted rescaling, by setting

$$\eta=xt^{-\beta}, \quad \tau=\log t, \quad v(\eta,\tau)=t^{\alpha}u(x,t), \tag{3.47}$$

after the transformation we arrive again to the equation (3.38). The following Lemma is standard and we omit its proof (see [62])

Lemma 3.1. *The profiles F_C can be characterized as the unique nonnegative stationary solutions of the equation* (3.38) *such that $f\in L^1(\mathbb{R}^n)$ and $f\in W^{1,p}(\mathbb{R}^n)$.*

We thus obtain that all the elements of $\omega(v)$ are of type F_C for some $C>0$, and the range of constants C is bounded above and below, using the corresponding sub- and supersolutions we have constructed. We still have to prove that there exists in the limit only one profile of this type, i.e. a unique constant C. This is again proved using mass arguments.

Mass analysis. In order to finish, we prove now the uniqueness of the limit profile. We already know that $\omega(v)=\{F_C: C_-\leq C\leq C_+\}$, these bounds coming from comparison with the subsolutions and the supersolutions constructed above. Define

$$m(\tau)=\int_{\mathbb{R}^n} v(\eta,\tau)d\eta. \tag{3.48}$$

Then $m(F_C)$ is increasing in C and $v(\tau_j) \to F_C$ uniformly for some subsequence $\tau_j \to \infty$ if and only if $m(\tau_j) \to m(F_C)$. We argue by contradiction and suppose that there exist $C_1 < C_2$ such that $m(\tau_j) \to m(F_{C_1})$ and $m(\tau_j^{'}) \to m(F_{C_2})$ on two subsequences $\tau_j \to \infty$ and $\tau_j^{'} \to \infty$ as $j \to \infty$. Then, for any $C \in (C_1, C_2)$, since $m(\tau)$ is bounded and continuous, there exists a subsequence $\tilde{\tau}_j \to \infty$ such that $m(\tilde{\tau}_j) \to C$ and $v(\tilde{\tau}_j) \to F_C$ uniformly in $\mathbb{R}^n$. Hence, all the points in (C_1, C_2) are limit points of $m(\tau)$ and this function has a very oscillatory character as $\tau \to \infty$. By a simple calculus fact, for any $C \in (C_1, C_2)$ we may suppose not only that $v(\tilde{\tau}_j) \to F_C$ uniformly, but also that

$$\frac{\partial}{\partial \tau} m(\tilde{\tau}_j) \leq 0, \tag{3.49}$$

by passing to a subsequence if necessary.

On the other hand, using (3.37), we calculate:

$$\begin{aligned}\frac{\partial}{\partial \tau} m(\tau) &= \int\limits_{\overline{\Omega}(\tau)} \Delta_p v \, d\eta + \tau^{-\frac{p-2}{p-1}} \int\limits_{\overline{\Omega}(\tau)} \left(\frac{1}{p-1} v - \frac{p-2}{(p-1)\tau^{1/(p-1)}} v \right) d\eta \\ &= \frac{1}{p-1} \tau^{-\frac{p-2}{p-1}} \left[\left(1 - \frac{p-2}{\tau^{1/(p-1)}} \right) m(\tau) + (p-1)\tau^{\frac{p-2}{p-1}} \int\limits_{\partial\overline{\Omega}(\tau)} |\nabla v|^{p-2} \nabla v \cdot \nu \, d\sigma(\eta) \right],\end{aligned} \tag{3.50}$$

where ν is the outward normal vector to the boundary of $\overline{\Omega}(\tau)$. Since the uniform limits of $v(\tau)$ along subsequences are only profiles from the family F_C, it is easy to see that

$$(p-1)\tau^{\frac{p-2}{p-1}} \int\limits_{\partial\overline{\Omega}(\tau)} |\nabla v|^{p-2} \nabla v \cdot \nu d\sigma(\eta) \to 0, \quad \text{as} \quad \tau \to \infty,$$

hence

$$\lim_{j \to \infty} \tilde{\tau}_j^{\frac{p-2}{p-1}} \frac{\partial}{\partial \tau} m(\tilde{\tau}_j) = \frac{1}{p-1} m(F_C) > 0 \tag{3.51}$$

in contradiction with (3.49). This contradiction shows that $\omega(v)$ contains only one element, i.e. $\omega(v) = \{F_C\}$ for some C depending only on the initial data u_0 and the domain Ω.

Since $\omega(v)$ has only one element F_C, we find that $v(\tau) \to F_C$ uniformly as $\tau \to \infty$, far from 0, i.e. in sets of the form $\{|\eta| \geq \delta\}$ Rephrasing the result in the initial variables, we obtain (3.44) far from the hole G, more precisely in sets of the form $\{|x| \geq \delta t^{\beta} (\log t)^{-(p-2)/p(p-1)}\}$, as stated. □

3.4 Case of low dimensions: $n < p$

3.4.1 Sub- and supersolutions. Size estimates

In this section we will construct appropriate sub- and supersolutions for our problem starting from the dipole profile that we have described before. Since from now on we will use only the dipole solutions (3.10)-(3.11), we will drop for simplicity the index 2 from the exponents α and β.

Supersolutions. We want to find a dipole solution D_λ such that at $t = t_0 > 0$ fixed, $D_\lambda(x, t_0) \geq u(x, t_0)$. But using (3.13) and (3.14), we obtain that a general rescaled profile satisfies

$$F_\lambda(\eta) \sim \lambda^{p-\frac{(p-2)(p-n)}{p-1}} \eta^{\frac{p-n}{p-1}} \to \infty \quad \text{as} \quad \lambda \to \infty, \tag{3.52}$$

and the convergence is uniform in compact sets far from the origin. On the other hand, the support of F_λ tends to the whole space as $\lambda \to \infty$. Hence, if we fix $t_0 > 0$, there exists $\lambda > 0$ sufficiently large such that

$$D_\lambda(x, t_0) \geq u(x, t_0), \quad \text{supp } u(\cdot, t_0) \subset \text{supp } D_\lambda(\cdot, t_0) \tag{3.53}$$

By well-known comparison arguments, from (3.53) we deduce that the inequality holds at any later time, i.e. $u(x, t) \leq D_\lambda(x, t)$, for all $x \in \Omega$ and $t \geq t_0$.

Subsolutions. This case is much more difficult, since the dipole does not vanish on the boundary of Ω. In order to construct a subsolution, we have to combine the dipole with another subsolution, using a similar technique as in [80]. We define:

$$D_{\lambda,\tau}(x, t) = D_\lambda(x, t + \tau)$$

and

$$H_\tau(x, t) = A(t)(t + \tau)^{-\alpha} \left(\left(\frac{|x|}{R_0} \right)^{(p-n)/(p-1)} - 1 - a \frac{(|x| - r_1)_+}{(t + \tau)^l} \right)_+,$$

where λ, τ, R_0, r_1, a, l are positive free parameters that have to be chosen. We choose $A(t) = 2K(t)$, where $K(t) = \max_x F_\lambda(x(t + \tau)^{-\beta})$. It may be checked (by direct calculation) that H_τ is indeed a subsolution of the p-Laplacian equation in the whole $\mathbb{R}^n$.

Denote by $R_1(t)$ the radius of the interface of H_τ and $R_2(t)$ the radius of the interface of $D_{\lambda,\tau}$. We want $R_2(t) > R_1(t)$ for t sufficiently large. We remark that $R_2(t) \sim (t + \tau)^\beta$ and $R_1(t)$ is a solution of the equation:

$$R_1(t)^{(p-n)/(p-1)}(t + \tau)^l = a R_0^{(p-n)/(p-1)}(R_1(t) - r_1), \tag{3.54}$$

hence, after an easy calculation, $R_1(t) \sim (t + \tau)^{l(n-1)/(p-1)}$. Since $n < p$, it suffices to choose $l < \beta$ in order to get $R_1(t) < R_2(t)$ for $t \geq t_0$ sufficiently large. Hence, for any $t \geq t_0$, there exists $r^*(t)$ such that $1 < r^*(t) < R_1(t) < R_2(t)$, such that the two subsolutions intersect at $|x| = r^*(t)$, with the correct angle of intersection (see Figure 3.2 below). Define:

$$V_{\lambda,\tau}(x, t) = \begin{cases} 0, & \text{if } r < R_0 \text{ or } r > R_2(t), \\ H_\tau(x, t), & \text{if } R_0 \leq |x| \leq r^*(t), \\ D_{\lambda,\tau}(x, t), & \text{if } r^*(t) \leq r \leq R_2(t), \end{cases} \tag{3.55}$$

which is a well-defined subsolution of the problem (3.1) for $t \geq t_0$ sufficiently large. The following lemma shows that this family of subsolutions has good properties.

Lemma 3.2. *There exists a choice of the parameters τ, λ, R_0, r_1, a, l and a time t_0 sufficiently large such that for $t \geq t_0$ we have*

$$V_{\lambda,\tau}(x,t) \leq u(x,t), \quad \forall\, x \in \Omega. \tag{3.56}$$

Proof. We show first that there exists a time t_0 such that we have comparison at $t = t_0$. Fix t_0 large such that $V_{\lambda,\tau}$ is a subsolution. Choose first $l < \beta$, for example $l = \frac{1}{2}\beta$, and (by increasing t_0 if necessary) choose R_0, r_1 such that the annulus $\overline{W_{R_0,r_1}(0)}$ is included in the interior of $\operatorname{supp} u(\cdot,t_0)$. Then we choose λ, that measure the height of the subsolution $V_{\lambda,\tau}$, such that $V_{\lambda,\tau}$ lies below u at time t_0. In order to choose the delay τ, we ask that

$$\operatorname{supp} V_{\lambda,\tau}(\cdot,t_0) = \overline{W_{R_0,R_2(t)}(0)} \subset \operatorname{Int}(\operatorname{supp} u(\cdot,t_0)).$$

Hence we want that $R_2(t_0) = \xi_+(t_0) - \varepsilon$, where

$$\xi_+(t_0) = \sup\{r > 0 : B(0,r) \subset \operatorname{supp} u(\cdot,t_0)\}.$$

But this implies a unique time τ for t_0 sufficiently large and $\varepsilon > 0$ small. Finally, in order to choose a, we impose that $R_1(t_0) < R_2(t_0)$, and this implies a limitation for the value of a.

We end the proof by showing that for any $t \geq t_0$, the inequality (3.56) holds. This follows from standard comparison arguments (the Strong Maximum Principle) applied starting from $t = t_0$ as initial time. The only thing we need to check is that the previous construction can be done for $t > t_0$, and the necessary and sufficient condition is that $R_1(t) < R_2(t)$, for any $t > t_0$. But this holds true for sufficiently large t_0, due to the asymptotic rates of $R_1(t)$ and $R_2(t)$ and the fact that $l < \beta$. $\square$

We illustrate how the comparison is performed in Figure 3.2 below.

3.4.2 Outer analysis I: Dipoles and the ω-limit

In this part we introduce the concept of ω-limit of a renormalized orbit of a solution of (3.1) and relate it to a family of dipole solutions, which are our candidates for the asymptotic profile. The proof of the convergence to a particular dipole is long and delicate and will be continued in the next two sections. We fix the similarity exponents α and β taking the values introduced in (3.11) for the dipole solutions. The main result is the following

Theorem 3.3. *Let $1 \leq n < p$ and suppose also that u_0 is compactly supported. Then there exists a constant $\lambda > 0$, depending on n, p and the initial data u_0, such that*

$$\lim_{t\to\infty} t^{-\alpha}|u(x,t) - D_\lambda(x,t)| = 0, \tag{3.57}$$

with uniform convergence in all sets of the form $\{x \in \Omega : |x| \geq \delta t^\beta\}$, $\delta > 0$.

The theorem will be proved using the technique of optimal barriers, also used in previous works like [61] for the porous medium equation or [88] for the Barenblatt equation of the

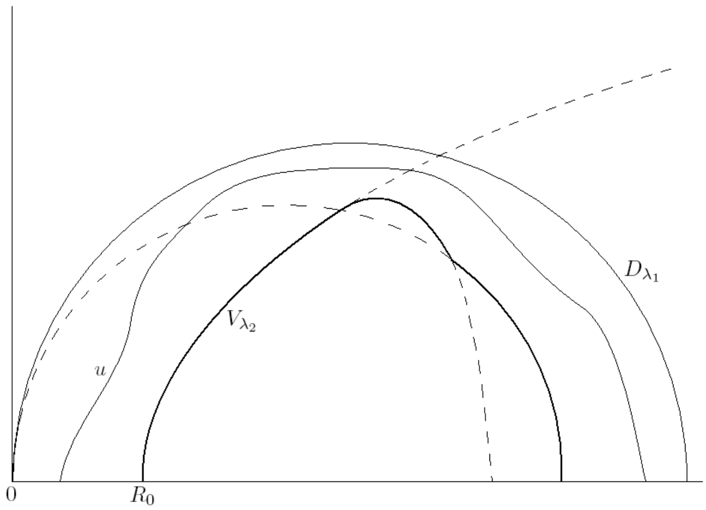

Figure 3.2: Comparison of the solution u with dipole profiles.

elasto-plastic filtration. The general idea of this technique is to construct the best barrier from above (or from below) for the asymptotic limit of the solution and then to show that in fact this optimal barrier equals the asymptotic limit, by using maximum and comparison principles. In the present case, the proof will be more involved than in these previous cases due to the degeneracy of the equation. This adds mathematical interest, specially in the analysis of contact points which can be seen as non-standard cases of application of the strong maximum principle in the sense of [120]. Two final observations: (i) the same result (3.57) is true for general L^1 data; this extension is easy and appears in our paper [81] (ii) in the present case of compactly supported solutions, we can also prove convergence of the free boundaries, see Corollary 3.2.

Let us proceed with the detailed proof. In the previous section, we have showed that there exist λ_1 and λ_2 such that

$$V_{\lambda_2,\tau}(x,t) \leq u(x,t) \leq D_{\lambda_1}(x,t+\tau), \quad \lambda_2 < \lambda_1. \tag{3.58}$$

This allows us to define

$$\lambda_1(t,\tau) = \inf\{\lambda_1 : u(x,t) \leq D_{\lambda_1}(x,t+\tau), \quad \forall\, x \in \Omega\}.$$

Comparison arguments for solutions imply that λ_1 is decreasing as a function of t. We may thus define the asymptotic limit $\lambda_1(\infty,\tau) = \lim_{t\to\infty} \lambda_1(t,\tau)$ and $\lambda_1(\infty,\tau) > 0$. In a similar way we may define

$$\lambda_2(t,\tau) = \sup\{\lambda_2 : V_{\lambda_2,\tau}(x,t) \leq u(x,t), \quad \forall\, x \in \Omega\},$$

hence there exists $\lambda_2(\infty,\tau) = \lim_{t\to\infty} \lambda_2(t,\tau)$ and it is easy to see that $\lambda_2(\infty,\tau) \leq \lambda_1(\infty,\tau)$, for any $\tau > 0$. The fact that the limit $\lambda_1(\infty,\tau)$ does not depend on the delay τ is a simple consequence of the following inequality satisfied by the dipole solutions.

Lemma 3.3. *For any $\varepsilon > 0$ small there exists $c(\varepsilon) > 0$ such that whenever $\lambda > 0$ and $|\tau| < c\,\varepsilon$:*

$$D_\lambda(x,t) \le D_{\lambda(1+\varepsilon)}(x, t(1+\tau)), \quad \forall\, x \in \mathbb{R}^n, \tag{3.59}$$

Proof. By scaling of D_λ we may reduce the proof to the case $t = 1$ and $\lambda = 1$. Let us examine the case $\tau > 0$. Away from $x = 0$ this is geometrically easy. By increasing the parameter $\lambda = 1$ to $1 + \varepsilon$, the maximal height and the radius of the support increase. By inserting the delay in time, it is easy to see, from the definition of D_λ, that the maximal height decreases and the radius of the support increases.

It remains to show that the inequality holds near $x = 0$. This follows from the estimate on the behavior of the dipole at $x = 0$ given in (3.14) and from the scaling formula (3.13). Indeed, in the first approximation near $x = 0$, we have:

$$D_1(x,t) \sim C_\lambda t^{-\alpha-\beta\frac{p-n}{p-1}} |x|^{\frac{p-n}{p-1}},$$

and

$$D_{1+\varepsilon}(x, t+\tau) \sim C_{1+\varepsilon}(t+\tau)^{-\alpha-\beta\frac{p-n}{p-1}}(1+\varepsilon)^{p-\frac{(p-2)(p-n)}{p-1}} |x|^{\frac{p-n}{p-1}}.$$

It is now easy to check that, fixing $t = 1$, there exists $\tau_1 = \tau_1(\varepsilon)$ such that the conclusion holds for any $0 < \tau < \tau_1$, and the relation between τ_1 and ε is linear for $\varepsilon \approx 0$. A similar argument near $x = 0$ will be used later to separate the contact in the origin.

The case $\tau < 0$ is geometrically easier and we leave it to the reader. □

3.4.3 Scalings, ω-limit and optimal bounds

For our next step we recall that the asymptotic analysis will depend on rescalings and limits. The rescaling that we will be using repeatedly is

$$(T_\gamma u)(x,t) = \gamma^\alpha u(\gamma^\beta x, \gamma t) \tag{3.60}$$

with exponents as in (3.11); in the sequel we often write $u_\gamma(x,t)$ instead of $(T_\gamma u)(x,t)$ for brevity. This rescaling keeps unchanged each of the dipole solutions D_λ, and when applied to a solution u, the whole family $\{T_\gamma u = u_\gamma\}$ consists of solutions of the p-Laplacian equation. Moreover, the inequality in Subsection 3.4.2 becomes

$$V_{\lambda_2,\tau/\gamma}(x,t) \le (T_\gamma u)(x,t) \le D_{\lambda_1}(x,t), \tag{3.61}$$

for all $t \ge t_0/\gamma$. From the compactness estimates in [52], we can extract a subsequence $\{T_{\gamma_k} u\}$ converging to a limit U_∞ as $\gamma_k \to \infty$; it is easy to see that this U_∞ is a local weak solution of the p-Laplacian equation in $\mathbb{R}^n \setminus \{0\} \times (0,\infty)$, and the convergence is uniform on compact subsets of $\mathbb{R}^n \setminus \{0\}$. The limit function U_∞ can (and will) have a singularity at $x = 0$, and there could in principle be different limits depending on the chosen subsequence.

Following dynamical systems terminology, we denote by $\omega(u)$ the ω-limit of the orbit $u(t)$, i.e., the set of all asymptotic limits of sequences u_{γ_k} as $\gamma_k \to \infty$. A generic element of

$\omega(u)$ will be denoted by U. Using (3.61) and the fact that through our rescaling, the delay disappears in the limit, we find that

$$D_{\lambda_2(\infty,\tau)}(x,t) \leq U(x,t) \leq D_{\lambda_1(\infty,\tau)}(x,t),$$

for all $U \in \omega(u)$, $\tau > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Hence, we can reduce τ to 0 in the previous inequality. It now follows from standard theory that $\omega(u)$ is a bounded, closed and connected set in the space of continuous functions $C(Q)$ for every $Q = \mathbb{R}^n \times [t_1, t_2]$ with $0 < t_1 < t_2 < \infty$.

With this in mind, we define the *optimal dipole parameter from above* as

$$\lambda^* = \inf\{\lambda > 0 : \; U(x,t) \leq D_\lambda(x,t) \text{ in } Q = \mathbb{R}^n \times (1/2, \infty), \text{ for all } U \in \omega(u)\}, \tag{3.62}$$

where we look at U and D_λ as extended by zero at the origin. Obviously, $\lambda_2(\infty, \tau) \leq \lambda^* \leq \lambda_1(\infty)$. In a similar manner, if we fix $U \in \omega(u)$, we can associate to it an upper optimal parameter λ_U defined as

$$\lambda_U = \inf\{\lambda > 0 : \; U(x,t) \leq D_\lambda(x,t) \text{ in } Q = \mathbb{R}^n \times (1/2, \infty)\}. \tag{3.63}$$

The pair (U, D_{λ_U}) will be called an *optimal pair*. It is obvious that $\lambda_U \leq \lambda^*$ for any $U \in \omega(u)$, moreover it is also easy to remark that

$$\lambda^* = \sup\{\lambda_U : \; U \in \omega(u)\}.$$

On the other hand, for any U there exists a unique optimal pair (U, D_λ), due to the fact that the family $\{D_\lambda\}$ is strictly increasing with respect to λ.

We will prove next a series of results in order to show that D_{λ^*} is the unique element of $\omega(u)$, which will end also the proof of Theorem 3.3. Let us remark first that, from the definition of λ^*, we have that $U \leq D_{\lambda^*}$ for any $U \in \omega(u)$.

3.4.4 Outer analysis II. Contact points and separation

In this section we analyze in detail the optimal pairs (U, D_{λ_U}) introduced in the previous section. As an intermediate step in our asymptotic analysis, we want to prove that $U = D_{\lambda_U}$. Arguing by contraction, if there is one $U \in \omega(u)$ that does not coincide with D_{λ_U}, then at least $U \leq D_{\lambda_U}$, and there could be three types of isolated contact points between U and D_{λ_U}. These are:

(a) Contact at a point $P = (x,t)$ which is not critical for D_{λ_U};

(b) Contact in the spatial maximum point (hot spot) of D_{λ_U}.

(c) Contact on the free boundary of the two functions;

We will refer to these types of contact points as contact points of type (a), (c), (b) respectively (see the sketch in Figure 3.3). In what follows we prove that all the three types of contact points stated above either imply exact equality or are impossible (disappear) after finite time.

Lemma 3.4. *A contact of type (a) implies equality.*

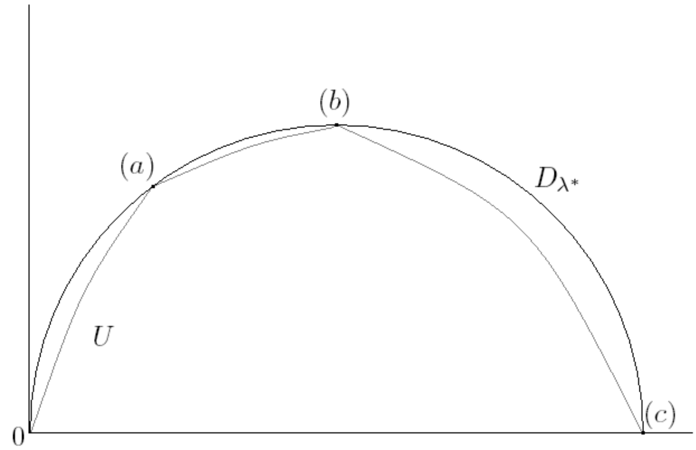

Figure 3.3: Contact points of types (a), (c) and (b).

Proof. Since the contact point is not a critical point for D_{λ_U}, this is an immediate consequence of the strong maximum principle for the p-Laplacian equation at nondegenerate points. We recall that weak solutions of the p-Laplacian evolution equation are $C^{1,\alpha}$ smooth with respect to the x variable, see [52]. □

3.4.5 Analysis of a type (b) contact. The strong maximum principle

In order to handle a contact point of type (b), where the equation degenerates for the solutions under consideration, we use the Harnack inequality proved by F. Chiarenza and R. Serapioni in [42] and improved in [43], for linear degenerate parabolic equations of the type

$$u_t = \mathrm{div}\,(a(x,t)\nabla u). \tag{3.64}$$

We recall that the result holds if the matrix $a(x,t)$ may be degenerate but it is controlled in terms of a Muckenhoupt weight, [44]. More precisely, it satisfies the following technical assumptions around some fixed point (x_0,t_0): there exists a non-negative function $\omega(x,t)$ defined on $\mathbb{R}^n \times (0,\infty)$ and some positive constant Γ such that

$$\omega(x,t)|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \le \Gamma\omega(x,t)|\xi|^2, \tag{3.65}$$

for a.e. $(x,t) \in Q$ and $\xi \in \mathbb{R}^n$, where $Q = \Omega \times (0,T)$, with $\Omega \subset \mathbb{R}^n$ open, and the function $\omega(x,t)$ is an A_2 weight in the time variable uniformly in x and an $A_{1+2/n}$ weight in x uniformly in time, i. e., it satisfies the two conditions:

$$\Big(\frac{1}{|B|}\int_B \omega(x,t)\,dx\Big)\Big(\frac{1}{|B|}\int_B \omega(x,t)^{-n/2}dx\Big)^{2/n} \le c_0, \quad \forall\, t > 0 \tag{3.66}$$

and

$$\Big(\frac{1}{|I|}\int\limits_I \omega(x,t)\,dt\Big)\Big(\frac{1}{|I|}\int\limits_I \omega(x,t)^{-1}dt\Big) \le c_0, \quad \forall\, x \tag{3.67}$$

for some $c_0 > 0$, where B represents any ball centered at (x_0, t_0) with sufficiently small radius and $I \subset (0,\infty)$ any small time interval. Thus, the Harnack inequality holds on some special cylinders, depending on the degeneracy of the operator around the point. The precise definitions of these cylinders are given in [43], Definition 3.2, and the Harnack inequality is proved as Theorem 3.4 of the same paper [43]. We apply this result to the analysis of our contact point.

Lemma 3.5. *A contact of type (b) is impossible to hold at any time $t > 0$ unless there is equality for all x and all later times.*

Proof. (i) *Linearization.* Remember that we are assuming that U and D_{λ_U} are not identically equal. Suppose that we have a contact of type (b), so that $\nabla U = \nabla D_{\lambda_U} = 0$ at (x_0, t_0). Set

$$w = U - D_{\lambda_U}, \tag{3.68}$$

which has an isolated zero at (x_0, t_0) and it is a solution of the linearized equation

$$w_t = \operatorname{div}(a(x,t)\nabla w), \tag{3.69}$$

where

$$a_{ij}(x,t) = \int\limits_0^1 |\nabla v(s)|^{p-4}((p-2)\partial_i v(s)\partial_j v(s) + |\nabla v(s)|^2 I_n)ds$$

is the matrix giving the degeneracy of the equation (3.69) in a parabolic neighbourhood $\mathcal{C}$ centered at (x_0, t_0), where we denote

$$v(s; x, t) = \nabla D_{\lambda_U} + s(\nabla U - \nabla D_{\lambda_U})$$

and I_n is the usual identity matrix. In the sequel we write $v(s)$ instead of $v(s; x, t)$. Since the matrix $\{(p-2)\partial_i v(s)\partial_j v(s)\}_{i,j}$ is positive definite, it is sufficient to bound from below the second term, i.e. to have a bound from below for $\int\limits_0^1 |\nabla v(s)|^{p-2}ds$. On the other hand, the bound from above comes from an obvious inequality, that is

$$\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \le (p-1)\int\limits_0^1 |\nabla v(s)|^{p-2}ds\,|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^n.$$

From these inequalities, we can take in (3.65)

$$\omega(x,t) = \int\limits_0^1 |\nabla v(s)|^{p-2}ds, \text{ and } \quad \Gamma = p-1.$$

for all $(x,t) \in \mathcal{C}$, and extended in a nondegenerate way outside of $\mathcal{C}$.

(ii) *Lower estimate.* We want to show that this weight satisfies in any case the conditions (3.66) and (3.67). First of all, we can bound it from below by terms depending only on estimates on $|\nabla D_{\lambda_U}|$, independent of the second term. At all points where $\nabla D_{\lambda_U} \neq 0$ we have:

$$\int_0^1 |\nabla D_{\lambda_U} + s(\nabla U - \nabla D_{\lambda_U})|^{p-2} ds = |\nabla D_{\lambda_U}|^{p-2} \int_0^1 |a+sb|^{p-2} ds,$$

where

$$a = \frac{\nabla D_{\lambda_U}}{|\nabla D_{\lambda_U}|}, \quad b = \frac{\nabla U - \nabla D_{\lambda_U}}{|\nabla D_{\lambda_U}|},$$

hence a is a unit vector. By performing a rotation if necessary, we may assume that $a = e_1$, the first vector of the canonical base of $\mathbb{R}^n$, hence we can work with scalar $a = 1$ and b. But it is easy to see that if we define

$$f(c) = \int_0^1 |1+sb|^{p-2} ds,$$

the function f admits a positive minimum as a function of b. We conclude that the matrix $\{a_{ij}(x,t)\}_{i,j}$ is bounded from below by $C|\nabla D_{\lambda_U}(x,t)|^{p-2}$ near (x_0,t_0). Hence the worse possible degeneracy order at (x_0,t_0) is given by the dipole solution $|\nabla D_{\lambda_U}(x,t)|^{p-2}$. We deduce that, in order to check the conditions (3.66) and (3.67) on $\omega(x,t)$, it is sufficient if they hold for $|\nabla D_{\lambda_U}|^{p-2}$.

On the other hand, using the fact that D_{λ_U} is a radial solution of the p-Laplacian equation and the correspondence relations between radial solutions of the p-Laplacian equation and the porous medium equation developed in [78], together with the behavior of self-similar profiles of the porous medium equation near a point of change of sign given in [75], we obtain that

$$|\nabla D_{\lambda_U}(x,t)| \sim C|x-x_0|^{1/(p-1)}$$

near (x_0,t_0), hence the maximal possible spatial degeneracy of (3.69) around (x_0,t_0) is like $|x-x_0|^{(p-2)/(p-1)}$.

(iii) *A_p conditions.* We are now ready to check the conditions. The only problem is the behavior of the last integrals near the line of degeneracy, $x(t) = x_0\,(t/t_0)^{\beta}$. It is easy to see that the maximal degeneracy with respect to the time variable near (x_0,t_0) is like

$$c|t^{\beta} - t_0^{\beta}|^{(p-2)/(p-1)} \sim ct_0^{\beta}\left|\left(\frac{t}{t_0}\right)^{\beta} - 1\right|^{(p-2)/(p-1)} \sim c\beta t_0^{\beta}(t-t_0)^{(p-2)/(p-1)}.$$

But a weight like $\Omega(x) = |x|^{(p-2)/(p-1)}$ satisfies (3.66) and a time weight like $c|t-t_0|^{(p-2)/(p-1)}$ satisfies (3.67). Thus, the Harnack inequality ([43], Theorem 3.4) applies and shows that $\inf_{\mathcal{C}} w > 0$, where $\mathcal{C}$ is a small special cylinder around (x_0,t_0), of type $\{|x-x_0| < r\} \times \{t_0 - k(x_0,t_0,r) < t < t_0\}$, see [43] for details, in particular there is no contact of type (b) between

U and D_{λ_U} at times before t_0. Hence, we can not have a contact of type (b) at all times (except in the trivial case when $U \equiv D_{\lambda_U}$, which we are assuming not to hold). Consequently, there exists a time $t_0 > 0$ such that we have no contact of type (b) at (x_0, t_0).

(iv) *Barrier argument.* We end the proof by showing that there is no contact of type (b) at times after t_0. Since we have no such contact at (x_0, t_0), there exists an annulus $r_1^0 < |x| < r_2^0$, containing the maximum points of D_{λ_U} at t_0 (i.e. with $|x| = |x_0|$), such that in this annulus we have a uniformly strict inequality $U(x, t_0) < D_{\lambda_U}(x, t_0)$. Consider $t \in [t_0, T]$, with $T < \infty$ arbitrary and denote by $r(t) = r_0\,(t/t_0)^\beta$ the absolute value of the spatial maximum points of $D_{\lambda_U}(\cdot, t)$. Let $0 < r_1(t) < r(t) < r_2(t)$ be such that $r_1(t_0) = r_1^0$, $r_2(t_0) = r_2^0$ and $r_i(t)$ continuous for $t_0 \le t \le T$. Since there is no contact of type (a), for $|x| = r_1(t)$ or $|x| = r_2(t)$, we have $U(x,t) < D_{\lambda_U}(x,t)$ uniformly. Since the application $\varepsilon \mapsto D_{\lambda_U - \varepsilon}$ is uniformly continuous, we find $\varepsilon > 0$ (depending on T) sufficiently small such that

$$D_{\lambda_U - \varepsilon}(x,t) > U(x,t),$$

for $|x| = r_i(t)$, $i = 1, 2$, $t_0 < t \le T$, and for $t = t_0$, $r_1^0 < |x| < r_2^0$, i.e., in a whole parabolic boundary of a domain in $\mathbb{R}^{N+1}$. Hence, this inequality extends to the interior at any time $t \in (t_0, T)$. In other words, $U \le D_{\lambda_U - \varepsilon}$ in the region $t_0 \le t \le T$, $r_1(t) < |x| < r_2(t)$, and consequently U lies strictly below D_{λ_U}. In particular, since T was arbitrarily large, this shows that a contact of type (b) is impossible after t_0. Note finally that we can take t_0 as small as we please. $\square$

3.4.6 Separation alternative

We continue here the effort to prove that every $U \in \omega(u)$ is in fact a dipole solution. The proof will depend on whether the strong maximum principle at points of type (b) is uniform in the following sense.

Lemma 3.6. *For any optimal pair (U, D_{λ_U}), with $U \in \omega(u)$, the following alternative holds: either we have asymptotic separation*

$$\inf_{t>1, |x|=|x_0(t)|} t^\alpha (D_{\lambda_U}(x, t+\tau_0) - U(x,t)) > 0, \tag{3.70}$$

or $D_{\lambda_U} \in \omega(U)$. Moreover, in that case $D_{\lambda_U} \in \omega(u)$

Proof. Suppose that the infimum in the statement is 0. Then, there exists a sequence $\{t_n\}$ of times such that

$$\lim_{n\to\infty} t_n^\alpha (D_{\lambda_U}(x, t_n + \tau_0) - U(x, t_n)) = 0.$$

Using the rescaling (3.60) with $\gamma = t_n^{-1}$, we find that there exists a sequence $U_n = U_{t_n^{-1}}$ of rescaled versions of U such that it converges to a limit U^* which touches D_{λ_U} at time $t = 1$ and $|x| = |x_0(1)|$ (the existence of the limit follows from classical compactness estimates, see [52]). But from Lemma 3.5 this is not possible, unless $U^* \equiv D_{\lambda_U}$. This proves the statement. The fact that $D_{\lambda_U} \in \omega(u)$ follows easily from a standard diagonal argument. $\square$

Assume now that the strong separation (3.70) does not hold. In this case, we prove:

Lemma 3.7. *If $D_{\lambda_U} \in \omega(U)$, then $U \equiv D_{\lambda_U}$. Consequently, any optimal pair reduces in this case to the dipole solution contained in it.*

Proof. Let $\{u_{\gamma_k}\}$ be a subsequence converging to U. We prove first that the family $u_{\gamma_k} = T_{\gamma_k} u$ becomes arbitrarily close to D_{λ_U} for k large, at time $t = 1$. This is our next claim:
Claim: *For any $\varepsilon > 0$, there exists $k = k(\varepsilon)$ sufficiently large such that $u_{\gamma_k} = T_{\gamma_k} u > D_{\lambda_U} - \varepsilon$ in $\Omega(\gamma_k) := \gamma_k^{-\beta}\Omega$, at time $t = 1$, for all $k \geq k(\varepsilon)$.*
Proof of the claim. Fix $t = 1$ and suppose that the claim is false, hence there exist $\varepsilon_0 > 0$, a sequence (k_n) going to infinity and $x_n \in \Omega$ such that $u_{\gamma_{k_n}}(x_n, 1) < D_{\lambda_U}(x_n, 1) - \varepsilon_0$. Using the standard compactness estimates and passing to the limit, we find that there exists $K \subset \mathbb{R}^n \setminus \{0\}$ compact set such that $U(x, 1) < D_{\lambda_U}(x, 1) - \varepsilon_0$, for all $x \in K$. On the other hand, from hypothesis, there exists a subsequence of rescaled versions of U converging to D_{λ_U}. Then, by uniform continuity of the map T_γ, there exists a first γ_0 such that $U_{\gamma_0}(x, 1) + \varepsilon_0 \geq D_{\lambda_U}(x, 1)$ in $\mathbb{R}^n$ and the two functions will touch. But their contact points are necessarily interior points for D_{λ_U}, since near the origin and near the free boundary $D_{\lambda_U} - \varepsilon_0 < 0$, and this is impossible from Lemmas 3.4 and 3.5 and the fact that $U_{\gamma_0} \in \omega(u)$.
Last argument. We choose k sufficiently large such that $D_{\lambda_U}(x, 1) \leq \varepsilon$, for all $x \in \partial\Omega(\gamma_k)$ and $k \geq k(\varepsilon)$. From (3.14), we deduce that $D_{\lambda_U}(x, t) \leq \varepsilon$ for all $x \in \mathbb{R}^n \setminus \Omega(\gamma_k)$ and for all $t > 1$. We compare then u_{γ_k} and $\widetilde{u} = D_{\lambda_U} - \varepsilon$ in $Q_k = \Omega(\gamma_k) \times [1, \infty)$, where k is large as in the previous step. Both are solutions for the original p-Laplacian equation (3.1). Moreover, the claim proved above gives us comparison at $t = 1$ for any $k \geq k(\varepsilon)$, and the discussion above shows that $\widetilde{u} \leq 0 = u_{\gamma_k}$ on $\partial\Omega(\gamma_k)$ for all t. It follows from the Maximum Principle applied to the original equation that $u_{\gamma_k}(x, t) \geq \widetilde{u}(x, t) = D_{\lambda_U}(x, t) - \varepsilon$, in Q_k and for all $k \geq k(\varepsilon)$. Passing to the limit in k, we obtain that $U(x, t) \geq D_{\lambda_U}(x, t) - \varepsilon$ in $Q_* = (\mathbb{R}^n \setminus \{0\}) \times [1, \infty)$, for all $U \in \omega(u)$. Since $U \leq D_{\lambda_U}$ and ε is arbitrarily small, we find that $U \equiv D_{\lambda_U}$, as desired. □

As an immediate consequence, we obtain that, if the strong separation assumption (3.70) does not hold, all the elements of $\omega(u)$ are necessarily dipole solutions with various parameters. In the next subsection we essentially treat the complementary case, where the strong separation holds.

3.4.7 The case of strong separation

We now study what happens if the separation assumption (3.70) holds.

Lemma 3.8. *Let (U, D_{λ_U}) be an optimal pair such that the strong separation assumption* (3.70) *holds. Then there exists $\widetilde{U} \in \omega(u)$ with $\lambda_{\widetilde{U}} < \lambda_U$.*

Proof. We start with the easier case where also the free boundaries of U and D_{λ_U} are separated. After, we show that the strong separation implies that we arrive to this situation in any case. We thus divide the proof into three steps.
Step 1. Assume for instance that the free boundaries of U and D_{λ_U} are separated at $t = t_0 > 0$. By rescaling we may assume that $t_0 = 1$. Using the separation Lemma 3.5, we

can take $\varepsilon > 0$ sufficiently small such that $U(x,1) \le D_{\lambda_U-\varepsilon}(x,1)$ for $|x| \ge x_0(t)$. We look for a small time advancement $\tau_1(\varepsilon) > 0$ such that $U(x,1) \le D_{\lambda_U-\varepsilon/2}(x,1-\tau_1)$ for all x near the origin. In order to find this τ_1, we recall the scaling (3.13) and the behavior of the dipole profiles near the origin given by (3.14) and it is enough to have $D_{\lambda_U-\varepsilon/2}(x,1-\tau_1) \ge D_{\lambda_U}(x,1)$ for all $x \in \mathbb{R}^n$ with $|x|$ sufficiently small. Comparing their principal terms, we need that

$$C_{\lambda_U-\varepsilon}(1-\tau_1)^{-\alpha-\beta\frac{p-n}{p-1}}(\lambda_U-\varepsilon/2)^{p-\frac{(p-2)(p-n)}{p-1}}|x|^{\frac{p-n}{p-1}} \ge C_{\lambda_U}(\lambda_U)^{p-\frac{(p-2)(p-n)}{p-1}}|x|^{\frac{p-n}{p-1}},$$

or, equivalently, that

$$(1-\tau_1)^{\alpha+\beta\frac{p-n}{p-1}} \le \frac{C_{\lambda_U-\varepsilon/2}}{C_{\lambda_U}}\Big(\frac{\lambda_U-\varepsilon/2}{\lambda_U}\Big)^{p-\frac{(p-2)(p-n)}{p-1}},$$

which is the condition that τ_1 should satisfy. By eventually decreasing ε, we find $\tau_1(\varepsilon) > 0$ sufficiently small such that the above condition is satisfied and the free boundaries of $U(x,1)$ and $D_{\lambda_U-\varepsilon/2}(x,1-\tau_1)$ are still separated. Then, we easily get that $U(x,1) \le D_{\lambda_U-\varepsilon/2}(x,1-\tau_1)$ for all $x \in \mathbb{R}^n$. By standard comparison, we then find

$$U(x,t) \le D_{\lambda_U-\varepsilon/2}(x,t-\tau_1), \quad \text{for } x \in \mathbb{R}^n, t > 1.$$

By rescaling, we obtain that

$$U_\gamma(x,t) \le D_{\lambda_U-\varepsilon/2}(x,t-\tau_1/\gamma), \quad \text{for } x \in \mathbb{R}^n, t > 1,$$

for any $\gamma > 0$. Passing to the limit in γ, we find in this case that $\omega(U)$ admits $D_{\lambda_U-\varepsilon/2}$ as upper bound. Since $\omega(U) \subseteq \omega(u)$, there exists an element $\widetilde{U} \in \omega(u)$ (in fact any element of $\omega(U)$ is good in this sense) such that $\lambda_{\widetilde{U}} < \lambda_U$.

Step 2. We will now assume that we are in the situation of a free boundary contact and that the strong separation assumption (3.70) holds. In that case we consider comparison of $U(x,t)$ and $D_{\lambda_U-\varepsilon}(x,t+\tau_0/2)$ for some $\varepsilon > 0$ in the region $Q_+ = \{(x,t) : t \ge 1, |x| \ge x_0(t)\}$. We choose $\varepsilon > 0$ sufficiently small such that $D_{\lambda_U-\varepsilon}(x,1+\tau_0/2) \ge U(x,1)$, ensuring in this way the comparison at our initial time $t = 1$. The comparison on the lateral boundary $|x| = x_0(t)$ follows from the strong separation (3.70). We conclude that

$$D_{\lambda_U-\varepsilon}(x,t+\tau_0/2) \ge U(x,t) \quad \text{in } Q_+,$$

hence their free boundaries are ordered,

$$r[U](t) \le r[D_{\lambda_U-\varepsilon}](t+\tau) = c(\lambda_U-\varepsilon)^\sigma(t+\tau)^\beta$$

and for large times we get separation of the free boundaries of U and D_{λ_U}, which leads us to the previous step.

Step 3. We conclude that the free boundary contact disappears if the separation assumption (3.70) holds, hence we can separate the free boundaries of U and D_{λ_U} for large times. After this, we arrive at the case in Step 1, hence the lemma is proved. □

We remark that in this case we can not conclude that $U \equiv D_{\lambda_U}$ directly, but the result of Lemma 3.8 will be used in the next subsection together with new arguments to arrive at such conclusion.

3.4.8 Outer analysis III: tail analysis and uniqueness

In this section, we prove that $\omega(u) = \{D_{\lambda^*}\}$, finalizing in this way the proof of Theorem 3.3. From the previous analysis, we know that $\omega(u)$ contains only dipole solutions or solutions bounded from above by such dipoles (as it comes from the strong separation alternative treated in the previous section). The main difficulty of proving that this set reduces in fact to a unique solution (for example the maximal one) is that the functions u_γ could have a long thin tail, i. e. a region where $|u_\gamma| \le \varepsilon$ very small, but that region could be a priori very large. The existence of such a tail makes difficult any comparison argument, since the supports of the rescaled functions may be much greater than the supports of their limits. Hence, the analysis we do is based on elimination or reduction of such a tail.

3.4.9 Bounds for the tail

In a first step we show that the tail is not larger in the limit than the support of the maximal dipole D_{λ^*}. Denote by $r_\gamma(t,\theta) = r[u_\gamma](t,\theta)$ the maximum free boundary radius of u_γ for fixed parameter γ, time t and angle $\theta \in \mathbb{S}^{n-1}$. Likewise, we let $R_\lambda(t) = r[D_\lambda](t) = c(\lambda,p,d)\,t^\beta$ be the maximum radius for D_λ, which does not depend on θ. We denote then by $C(t,\theta) := \limsup_{\gamma\to\infty} r_\gamma(t,\theta)$. With these notations, we prove:

Lemma 3.9. *For any $t > 0$ and $\theta \in \mathbb{S}^{n-1}$, we have that $C(t,\theta) \le R_{\lambda^*}(t) = c_* t^\beta$.*

Proof. (i) *A preliminary consequence of scaling.* We first prove that for any $t > 0$ and $\theta \in \mathbb{S}^{n-1}$ fixed, we have $C(t,\theta) = C(1,\theta)t^\beta$. We can write $u_\gamma(x,t) = t^{-\alpha}u_{\gamma t}(xt^{-\beta},1)$. Since the same family appears in both members, passing to maximal radii, we obtain

$$r_\gamma(t,\theta) = r_{\gamma t}(1,\theta)\,t^\beta,$$

hence, by taking limits, we find $C(t,\theta) = t^\beta C(1,\theta)$. Below we write $C(1,\theta) = C$ for brevity.
(ii) *Argument by contradiction.* Suppose that the statement is false and there exists $t_0 > 0$ and $\theta_0 \in \mathbb{S}^{n-1}$ such that $C(t_0,\theta_0) > R_{\lambda^*}(t_0)$. By the rescaling (i) we may assume that $t_0 = 1/2$. For simplicity, we take the direction θ_0 of maximum $C(1/2,\theta_0)$ that we consider fixed from now on and write $C(t) = C(t,\theta_0)$ and $r_\gamma(t) = r_\gamma(t,\theta_0)$. The plan of the argument is to show that at the time $t = 1$, we have $C(1) \le R_{\lambda^*}(1)$, which would contradict the original assumption in view of the power-like formulas $C(t,\theta) = C(1,\theta)t^\beta$ and $R_{\lambda^*}(t) = c_* t^\beta$.

Arguing by contradiction, suppose that $C(1) > R_{\lambda^*}(1)$. Passing to a subsequence, it follows that, for any $\varepsilon > 0$, there exists $k = k(\varepsilon)$ and a subsequence u_{γ_k} such that $r_{\gamma_k}(1) \ge C(1) - \varepsilon > R_{\lambda^*}(1)$, for all $k \ge k(\varepsilon)$.

Some notations: From part (i), it is immediate that $C(1/2) = C(1)/2^\beta < C(1)$. By eventually decreasing $\varepsilon > 0$, we may assume that $C(1) - 3\varepsilon > C(1/2)$ and at the same time $C(1) - 3\varepsilon > R_{\lambda^*}(1)$. Set $C_0 := \max\{C(1/2), R_{\lambda^*}(1)\} < C(1) - 3\varepsilon$. We are now ready for the main calculation.
(iii) *Comparison with a traveling wave.* Since the extra-part of the supports of u_γ takes the form of a thin tail, coming back to the subsequence γ_k chosen above, we may assume that for

$|x| \geq R_{\lambda^*}(t) + \varepsilon$, $t \in [1/2, 1]$ and $k \geq k(\varepsilon)$ large, we have $|u_{\gamma_k}(x,t)| \leq \varepsilon$. In order to control the length of this tail region, we consider the traveling wave

$$\hat{u}(x,t) = \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (\varepsilon(t-1/2) + \varepsilon + C_0 - x_1)_+^{(p-1)/(p-2)} \tag{3.71}$$

and we compare it with u_{γ_k} in the region $\{x_1 \geq C_0, t \in [1/2, 1]\}$, for $k \geq k(\varepsilon)$. Following the lines of proof of Theorem 18.8 in [131] and rotating the argument, we find that $r_{\gamma_k}(1) \leq C_0 + 2\varepsilon < C(1) - \varepsilon$, contradiction. Hence the supposition made in part (ii) is false and $C(1) \leq R_{\lambda^*}(1)$. $\square$

As an immediate consequence, if $\{u_{\gamma_l}\}$ is a subsequence converging to D_{λ^*} (if such a subsequence exists), we obtain that $r_{\gamma_l}(t,\theta) \to R_{\lambda^*}(t)$, since the estimate from below follows immediately from the locally uniform convergence. We need a convergence result for the free boundary under more general circumstances.

Lemma 3.10. *Let $\{u_{\gamma_k}\}$ be a subsequence converging to $U \in \omega(u)$ with (U, D_{λ_U}) optimal pair, $\lambda_U < \lambda^*$. Then, for all $t > 0$ and $\theta \in \mathbb{S}^{n-1}$,*

$$\lim_{k\to\infty} r_{\gamma_k}(t,\theta) = R_{\lambda^*}(t).$$

Proof. (i) Suppose that convergence from below is false and there exist $t_0 > 0$, $\gamma_0 > 0$ and some small $\varepsilon > 0$ and $\tau_0 \geq 0$ such that

$$u_{\gamma_0}(x,t_0) \leq D_{\lambda^*-\varepsilon}(x, t_0 - \tau_0), \ x \in \Omega(\gamma_0).$$

By parabolic comparison between the solutions u_{γ_0} and $D_{\lambda_*-\varepsilon}$, we find that $u_{\gamma_0}(x,t) \leq D_{\lambda^*-\varepsilon}(x, t-\tau_0)$, for any $t > t_0$, and by inverting the scaling, we have $u(x,t) \leq D_{\lambda^*-\varepsilon}(x, t-\tau_0\gamma_0)$, for any $t > t_0\gamma_0$. It follows that

$$u_\gamma(x,t) \leq D_{\lambda^*-\varepsilon}(x, t - \tau_0\gamma_0/\gamma),$$

for any $\gamma > 0$ and $t > t_0\gamma_0/\gamma$. This contradicts the definition of λ^*, since any limit $U \in \omega(u)$ is bounded above by $D_{\lambda^*-\varepsilon}$.
(ii) From the uniform convergence to U, for any $\delta > 0$, there exists $k = k(\delta)$ large such that $u_{\gamma_k} < D_{\lambda_U} + \delta$ on $\mathrm{supp}(D_{\lambda_U}) \cap \Omega(\gamma_k)$, for any $k \geq k(\delta)$. Hence, if

$$\liminf_{k\to\infty} r_{\gamma_k}(t,\theta) < R_{\lambda^*}(t),$$

then the situation in the previous paragraph can be obtained for some k very large (corresponding to δ small enough). Hence, the limit above should be at least $R_{\lambda^*}(t)$. Using also Lemma 3.9, we obtain that the limit is precisely $R_{\lambda^*}(t)$, for any $t > 0$. $\square$

3.4.10 Uniqueness of the limit profile. Final argument

We can now show that D_{λ^*} is the unique asymptotic limit. This will be a consequence of the following

Lemma 3.11. *Let (U, D_{λ_U}) be an optimal pair. Then necessarily $\lambda_U = \lambda^*$.*

Proof. Suppose not and consider an optimal pair (U, D_{λ_U}) with $\lambda_U < \lambda^*$. Then, there exists a subsequence u_{γ_k} converging to U. Now, we retake the technique of Lemma 3.9 and we want to compare the solutions u_{γ_k} with a similar traveling wave as in (3.71). In this case, we consider $t = 1$ as starting time, $t = 2$ as final time and define $C_0 := \max\{R_{\lambda^*}(1), R_{\lambda_U}(2)\}$. Choose $\varepsilon > 0$ so small such that $C_0 < R_{\lambda^*}(2) - 3\varepsilon$. The thin tail exists now at least for $|x| \geq R_{\lambda_U}(t) + \varepsilon$, k large, $t \in [1, 2]$, and in this region we may assume that $|u_{\gamma_k}(x, t)| \leq \varepsilon$. Then we define $\hat{u}$ as in (3.71), with our new C_0 and ε, and we compare u_{γ_k} and $\hat{u}$, for $k \geq k(\varepsilon)$ sufficiently large. By a similar comparison as in the proof of Lemma 3.9, we find that $r_{\gamma_k}(2, \theta) \leq C_0 + 2\varepsilon < R_{\lambda^*}(2) - \varepsilon$, for any $k \geq k(\varepsilon)$ sufficiently large and $\theta \in \mathbb{S}^{n-1}$. In conclusion, if there exists a subsequence converging to a limit U bounded above by a dipole with parameter $\lambda_U < \lambda^*$, we are able, after a time, to decrease the tail (uniformly in θ) with respect to the free boundary of D_{λ^*}. But this is a contradiction with the result of Lemma 3.10. $\square$

Corollary 3.1. *The strong separation alternative obtained in Subsection 3.4.6 is impossible.*

Proof. If we have strong separation between U and $D_{\lambda_U} = D_{\lambda^*}$, from Lemma 3.8, there exists an optimal pair $(\widetilde{U}, D_{\lambda_{\widetilde{U}}})$ with parameter $\lambda_{\widetilde{U}} \leq \lambda^* - \varepsilon/2$. But this is a contradiction with Lemma 3.11. $\square$

It follows that necessarily $\omega(u) = \{D_{\lambda^*}\}$ and Theorem 3.3 is finally proved. We also obtain, as an immediate consequence, that

$$\lim_{\gamma \to \infty} r_\gamma(t, \theta) = R_{\lambda^*}(t), \tag{3.72}$$

for any $t > 0$ and uniformly in $\theta \in \mathbb{S}^{n-1}$. This implies the convergence of supports and interfaces of the general solution u to those of D_{λ^*}. Indeed, if we introduce the following notations:

$$r_+(t) = \max_{x \in \Gamma(t)} |x|, \quad r_-(t) = \min_{x \in \Gamma(t)} |x|, \tag{3.73}$$

where $\Gamma(t)$ is the free boundary of the solution u at time t, then from (3.72) and usual rescaling, we can state the following:

Corollary 3.2. *In the conditions of Theorem* 3.3 *and with notations of the previous paragraphs, we have:*

$$\lim_{t \to \infty} \frac{r_\pm(t)}{R_{\lambda^*}(t)} = 1. \tag{3.74}$$

Remark: Since $F(\eta) \sim \eta^{(p-n)/(p-1)}$, we see that the dipole solution is a local weak solution of the p-Laplacian evolution equation in $\mathbb{R}^n \setminus \{0\} \times (0, \infty)$ with $n < p$, in the sense specified in Definition 3.2, but it is not a weak solution in the sense of Definition 3.1. Indeed, from the flux condition

$$\lim_{\eta \to 0} \eta^{n-1} |F^{'}(\eta)|^{p-1} = 0,$$

we obtain that, if $F(\eta) \sim \eta^{\gamma}$ as $\eta \to 0$, then the self-similar solution whose profile is $F(\eta)$ is a weak solution for $\gamma > (p-n)/(p-1)$. This fact shows that the singularity at $x = 0$ of the limit function can not be removed for $n < p$ (as it happens for $n > p$, see [80]).

3.5 Comments and open problems

1. Discussion about the inner behavior. By inner behavior, we understand the behavior of the solution u of the PLE near the holes (in bounded subdomains). We are able to describe the inner asymptotic behavior only in the simplest case, $n > p$, which is related to the unique solution of the following exterior Dirichlet problem:

$$\begin{cases} \Delta_p H = 0 \ \text{ in } \ \Omega, \\ H = 0 \ \text{ on } \ \partial\Omega, \\ H \to 1 \ \text{ uniformly } \ \text{as} \ |x| \to \infty, \end{cases}$$

by multiplying it by a constant $C > 0$. To find this constant we use the technique of matched asymptotics, together with the convergence of the Steklov averages, following the same ideas as in the paper [37]. The precise result is the following:

Theorem. *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a sufficiently large time $t_{in} = t_{in}(\varepsilon, \delta)$ such that*

$$\left| t^{\alpha} u(x,t) - C_0^{\frac{p-1}{p-2}} H_p(x) \right| \le \varepsilon, \tag{3.75}$$

for all $t \ge t_{in}$ and for all $x \in \Omega$ with $|x| \le \delta t^{\beta}$.

For the proof, we follow closely the techniques in [37], together with the elliptic estimate

Lemma. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in C(\Omega) \cap L^{\infty}(\Omega)$ and $u \in C^1(\Omega) \cap C(\overline{\Omega})$ be the solution of the Dirichlet problem:*

$$\begin{cases} \Delta_p u = f \quad in \ \ \Omega, \\ u = 0 \quad on \ \ \partial\Omega. \end{cases} \tag{3.76}$$

Then there exists a constant $C > 0$, independent on the diameter of Ω, such that

$$|u| \le C d^{\frac{p}{p-1}} (\sup_{\Omega} |f|)^{\frac{1}{p-1}} \quad in \ \ \Omega, \tag{3.77}$$

where $d = diam(\Omega)$.

which is an extension of the similar result for the Poisson equation presented as Theorem 3.7 in the book [66]. The complete, technical proof can be found in [80].

In the cases $n = p$ and $n < p$, we leave the inner asymptotic behavior still open. By using similar techniques, we only can obtain the following two incomplete results:

Proposition. *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a time $t_{in} = t_{in}(\varepsilon, \delta)$ sufficiently large, such that*

$$\left|(t\log t)^{\frac{1}{p-1}}u(x,t) - \frac{C_0^{(p-1)/(p-2)}H_p(x)}{\beta\log t}\right| \leq \varepsilon, \tag{3.78}$$

for all $t \geq t_{in}$ and for all $x \in \Omega$ with $|x| \leq \delta t^{\beta}(\log t)^{-(p-2)/p(p-1)}$.

for the resonant case $n = p$ and

Proposition. *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a sufficiently large time $t_{in} = t_{in}(\varepsilon, \delta)$ such that*

$$\left|t^{\alpha}u(x,t) - \frac{C_{\lambda_0}H_p(x)}{t^{\beta(p-n)/(p-1)}}\right| \leq \varepsilon, \tag{3.79}$$

for all $t \geq t_{in}$ and $x \in \Omega$ with $|x| \leq \delta t^{\beta}$.

in dimension $n < p$. These results only show that the decay in time of the solution in the inner region is strictly bigger than the decay in the outer region. That is a new phenomenon, and it happens because the hole plays a more important role and the mass is lost faster than in the outer region.

The inner asymptotic behavior for the Heat Equation in dimension $n = 2$ (the critical one in the case of the Laplacian) has been described in [73], and it consists in three regions: the outer region with the expected decay, an intermediate region where there are different scalings, and which can be seen as a transition region, and the inner region where the time decay is bigger (has an extra logarithmic term). We think that this complicated behavior will also happen for the p-Laplacian equation in dimension $n = p$ and in dimension $n < p$, but we still do not have a proof for this. Let us also mention that the inner behavior for the PME in dimensions $n = 1, 2$ and even for the heat equation in dimension $n = 1$ are still missing in literature (up to my knowledge).

2. Non-connected domains. The assumption of connectedness is not an essential restriction. If the domain is not connected, every connected component is treated separately. The bounded connected components follow the behavior of bounded domains described in [131], Chapter 20. In the unique unbounded component, the behavior is described by the results in the present chapter.

3. Open problem: quantitative estimates for the mass. A precise estimate of the mass $M(t)$ at any moment of time is still missing. In the porous medium case, due to a conservation law, a precise estimate was obtained in [37], Corollary 4.2. In our case, there seems to be no conservation law, and this makes more difficult to obtain a relation between the initial mass of the solution and $M(t)$.

4. The problem of hot-spots. The hot spots are the spatial maxima of the solution at any moment of time; more precisely,

$$\mathcal{H}(t) = \{x \in \overline{\Omega} : u(x,t) = \max_{y \in \overline{\Omega}} u(y,t)\}.$$

An interesting extension of the work presented in the previous sections would be to study the evolution of $\mathcal{H}(t)$. Only the linear case $p = 2$ is known, due to Ishige [84], and only in the particular case of the exterior of a ball. If we denote

$$\mathcal{H}_+(t) = \sup_{x \in \mathcal{H}(t)} |x|, \quad \mathcal{H}_-(t) = \inf_{x \in \mathcal{H}(t)} |x|,$$

for $\Omega = \mathbb{R}^n \setminus B(0,R)$, Ishige proves that

$$\begin{cases} \lim\limits_{t\to\infty} \frac{\mathcal{H}_\pm(t)}{t^{1/n}} = C(n,R), & \text{if } \; n \geq 3, \\ \lim\limits_{t\to\infty} \frac{\mathcal{H}_\pm(t)}{(t\log t)^{1/2}} = \sqrt{2}, & \text{if } \; n = 2, \end{cases}$$

where $C(n,R) = (2(n-2)R^{n-2})^{1/n}$. In the nonlinear case $p > 2$, our asymptotic results imply that the hot spots lie in some overlapping region, which is a region in Ω (whose boundaries evolve with different time scales) where both the outer and the inner behavior take place. We prove the existence of such a region in [80]; in the PME case in dimension $n > 2$ there exists a similar overlapping region, as shown in [37], but the precise behavior of the hot-spots is an open problem.

5. Comparison to the porous medium case. The results obtained for the p-Laplacian case are parallel to those for the porous medium equation in exterior domain, obtained previously in [37] and [69]. This should not be surprising at all after reading Chapter 2. Apart from the important differences in the techniques, some qualitative difference appears in low dimensions, where in the porous medium case there is no anomalous phenomenon. The unique subcritical dimension which makes sense physically for the PME is $n = 1$, although in the radially symmetric setting any positive dimension is allowed. The outer analysis for dimensions $n \in (1,2)$ was performed by Gilding and Gonzerkiewicz in [69], using a very different technique, based on comparison principles associated to some weighted integrals of the solutions (that we do not have in our problem). The asymptotic behavior is given by a dipole solution of the porous medium equation, having the general form:

$$Z_C(x,t) = t^{-\alpha} U(xt^{-\beta}), \quad U(\eta) = \pm|\eta|^{\frac{2-n}{m}} \left(C - \frac{m-1}{2(n(m-1)+2)} |\eta|^{n+\frac{2-n}{m}}\right)_+^{\frac{1}{m-1}}, \tag{3.80}$$

where $\alpha = 1/m$ and $\beta = 1/2m$ do not depend on the dimension $1 \leq n < 2$. The form of the profile depends on n, but it is explicit in all cases. So there is no anomalous phenomenon.

A different result concerning the porous medium equation can be obtained from our study using the correspondence relations between the p-Laplacian equation and the porous medium equation in [78]. Recall also that in [24], another family of self-similar solutions of the porous medium equation, denoted by $U_{3,\lambda}$, which have lap number 2, is studied, and it is proved that these solutions are anomalous. Here λ comes from a scaling similar to (3.13), with the precise formula

$$U_{3,\lambda}(x,t) = \lambda^2 U_3(\lambda^{1-m}x,t).$$

We deal here with the exterior Neumann problem in a half line. More precisely:

Proposition. *Consider, in dimension $n = 1$, the solution v of the following exterior Neumann problem:*

$$\begin{cases} v_t = (|v|^{m-1}v)_{xx}, & \text{in } \ \Omega \times (0,\infty), \\ v(x,0) = v_0(x), & \forall\, x \in \Omega, \\ v_x(0,t) = 0, & \forall\, t > 0, \end{cases}$$

where $\Omega = (0,\infty)$ and v_0 is a continuous function with only one change of sign in the positive x axis and with zero total mass, i.e. $\int_{\mathbb{R}} v_0(x)dx = 0$. Then there exists $\lambda > 0$ such that

$$t^{\alpha_3}(v(x,t) - U_{3,\lambda}(x,t)) \to 0 \quad as \quad t \to \infty$$

with convergence in $L^1(\mathbb{R})$.

The proof is an immediate consequence of the results of the present chapter and the fact that the solutions of the porous medium equation in dimension $n = 1$ may be obtained from those of the p-Laplacian equation in dimension $n = 1$ by differentiation cf. Chapter 2.

Chapter 4

Local smoothing effects, positivity estimates and Harnack inequalities for the fast p-Laplacian equation

4.1 Introduction

In this chapter we study the behavior of local weak solutions of the parabolic p-Laplacian equation

$$\partial_t u = \nabla \cdot (|\nabla u|^{p-2} \nabla u) \tag{4.1}$$

in the range of exponents $1 < p < 2$, which is known as the fast diffusion range. We consider weak solutions $u = u(x,t)$ defined in a space-time subdomain of $\mathbb{R}^{n+1}$ which we usually take to be, without loss of generality in the results, a cylinder $Q_T = \Omega \times (0,T]$, where Ω is a domain in $\mathbb{R}^n$, $n \geq 1$, and $0 < T \leq \infty$. The main goal of the present study is to establish *local upper and lower bounds* for the nonnegative weak solutions of this equation. By local estimates we mean estimates that hold in any compact subdomain of Q_T with bounds that do not depend on the possible behavior of the solution u near $\partial\Omega$ for $0 \leq t \leq T$. Our estimates cover the whole range $1 < p < 2$. The upper estimates extend to signed weak solutions as estimates in $L^\infty_{loc}(Q_T)$.

It is well-known that fast diffusion equations, like the previous one and other similar equations, admit local estimates, and there are a number of partial results in the literature. For the closely related fast-diffusion equation, $\partial_t u = \Delta(u^m)$ with $0 < m < 1$, interesting local bounds were found recently by two of the authors in [34], including the subcritical case $m < m_c := (n-2)/n$, where these estimates were completely new. On the other hand, the theories of the porous medium/fast diffusion equation and the p-Laplacian equation have strong similarities both from the quantitative and the qualitative point of view. This similarity is made explicit by the transformation described in [78] that establishes complete equivalence of the classes of radially symmetric solutions of both families of equations (note that the transformation maps m into $p = m + 1$ and may change the space dimension). However, the particular details of both theories for general non-radial solutions can be quite different, and the purpose of this study is to make a complete analysis of the issue for the p-Laplacian equation.

Let us mention that our parabolic p-Laplacian equation has been widely researched for values of $p > 2$, cf. [52] and its references, but the fast-diffusion range has been less studied, see also [53, 57, 55]. However, just as it happens to the fast diffusion equation for values of $m \sim 0$, the theory becomes difficult for p near 1, more precisely for $1 < p < p_c = 2n/(n+1)$, and such a low range is almost absent from the literature. For the natural occurrence of the exponent p_c in the theory see for instance [59] or the book [127], Chapter 11.

Some local estimates were established by Di Benedetto and Herrero in [53]. We will establish here new upper and lower bounds of local type, completing in this way these previous results, and setting a new basis for the qualitative study of the equation in that range.

A consequence of our local bounds from above and below is a number of *Harnack inequalities*. The question of proving Harnack inequalities for the fast p-Laplacian equation has been raised first by DiBenedetto and Kwong in [54]. This problem has been studied again recently by DiBenedetto, Gianazza and Vespri in [55], where they prove that the standard intrinsic Harnack inequality holds for $p > p_c$ and is in general false for $p < p_c$, and they leave as an open question the existence of Harnack inequalities of some new form in that low range of p. We give a positive answer to this intriguing open problem.

We also prove existence and sharp space-time asymptotic estimates for the so-called large solutions u_∞, namely, $u_\infty \sim t^{1/(2-p)}\mathrm{dist}(x,\partial\Omega)^{p/(p-2)}$, for any $1 < p < 2$. Moreover, we prove a new local energy inequality for suitable norms of the gradients of the solutions, which can be extended to more general operators of p-Laplacian type. As a consequence, we obtain that bounded local weak solutions are indeed local strong solutions, more precisely $\partial_t u \in L^2_{\mathrm{loc}}$, cf. Corollary 4.1. This qualitative information adds an important item to the general theory of the p-Laplacian type diffusions.

Some of the results and techniques may be also extended to more general degenerate diffusion equations, as mentioned in the concluding remarks.

Organization of the chapter. We begin with a section where we state the definitions and the main results of the present chapter in a concentrated form. It contains: local upper bounds for solutions, positivity estimates, Harnack inequalities and local inequalities for the energy, i. e., for the gradients of the solutions. The rest of the chapter will be divided into several parts, as follows:

Local Smoothing Effect for L^r norms. In Section 4.3, we give the proof of Theorem 4.1, which is the main Local Smoothing Effect. It is proved in a first step for the class of bounded local strong solutions. The proof (Subsection 4.3.3) is obtained by joining a space-time local smoothing effect (Subsection 4.3.1) with an L^r_{loc} stability estimate, i. e., we control the evolution in time of the local L^r norms, $r \ge 1$ (Subsection 4.3.2). The local smoothing result for general local strong solutions will be postponed to Section 4.5.

Let us point out here that as a consequence of this result and known regularity theory (cf. [52] or Appendix A2), it follows that the local strong solutions are Hölder continuous, whenever their initial trace lies in L^r_{loc} for suitable r.

Continuous Large Solutions. In Section 4.4, we apply the boundedness result of Theorem 4.1, to prove the existence of the so-called *large solutions* for the parabolic p-Laplacian equation for any $1 < p < 2$. We derive some of their properties, in particular we prove a sharp asymptotic behavior for large times. We also construct the so-called extended large solutions, in the spirit of [40]. These results are a key tool in the proof of our sharp local smoothing effect, when passing from bounded to general local strong solutions. Roughly speaking, extended large solutions play the role of (quasi) "absolute upper bounds" for local solutions.

Local lower bounds. We devote Sections 4.6 and 4.7 to establish lower estimates for local weak solutions, in the form of quantitative positivity estimates for small times, see Theorem 4.2, and estimates which are global in time, of the *Aronson-Caffarelli type*, see Theorem 4.3. In Section 4.6 we prove all these facts for a minimal Dirichlet problem, while in Section 4.7 we extend them to general continuous local weak solutions via a technique of local comparison.

Harnack inequalities. In Section 4.8, we prove forward, backward and elliptic Harnack inequalities in its intrinsic form, cf. Theorem 4.6, together with some other alternative forms, that avoid the delicate intrinsic geometry. This inequalities are sharp and extend to the very fast diffusion range $1 < p \le p_c$, the results of [54, 55] valid only in the supercritical range $p_c < p < 2$, for which we give a different proof.

A special energy inequality. In Section 4.9, we prove a new estimate for gradients, Theorem 4.7, which, besides its application in the proof of the Local Smoothing Effect, has several applications outlined in that section, such as the fact that bounded local weak solutions are indeed local strong solutions, cf. Corollary 4.1. This inequality can be extended to more general operators of p-Laplacian type. Let us also mention that such a technical tool is not needed in developing the corresponding theory for the fast diffusion equation.

Panorama, open problems and existing literature. In the last section we draw a panorama of the obtained results, we pose some open problems and we briefly compare our results with other related works.

4.2 Statements of the main results

4.2.1 The notion of solution

We use the following definition of local weak solution, found in the literature, cf. [52, 57].

Definition 4.1. *A "local weak solution" of* (4.1) *in* Q_T *is a measurable function*

$$u \in C_{loc}\big(0,T;L^2_{loc}(\Omega)\big) \cap L^p_{loc}\big(0,T;W^{1,p}_{loc}(\Omega)\big)$$

such that, for every open bounded subset K *of* Ω *and for every time interval* $[t_1,t_2] \subset (0,T]$, *the following equality holds true:*

$$\int_K u(t_2)\varphi(t_2)\,\mathrm{d}x - \int_K u(t_1)\varphi(t_1)\,\mathrm{d}x + \int_{t_1}^{t_2}\int_K \left(-u\varphi_t + |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\right)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad (4.2)$$

for any test function $\varphi \in W^{1,2}_{loc}\big(0,T;L^2(K)\big) \cap L^p_{loc}\big(0,T;W^{1,p}_0(K)\big)$. *Under similar assumptions, we say that* u *is a local weak subsolution (supersolution) if we replace in* (4.2) *the equality by* $\le$ *(resp.* $\ge$*) and we restrict the class of test functions to* $\varphi \ge 0$.

A local weak solution u *is called "local strong solution" if* $u_t \in L^1_{loc}(Q_T)$, $\Delta_p u \in L^1_{loc}(Q_T)$ *and equation* (4.1) *is satisfied for a. e.* $(x,t) \in Q_T$. *In the definition of local strong sub- or super-solution we only add the condition* $u_t \in L^1_{loc}(Q_T)$, *while the requirement* $\Delta_p u \in L^1_{loc}(Q_T)$ *is not imposed (and is in general not true).*

We will recall in the sequel known properties of the local weak or strong solutions at the point where we need them. We just want to stress the local (in space-time) character of the definition, since there is no reference to any initial and/or boundary data taken by the local weak solution u. However, *in some statements initial data are taken as initial traces in some space* $L^r_{\rm loc}(\Omega)$, and then $u \in C\big([0,T];L^r_{loc}(\Omega)\big)$. This can be done in view of the results of DiBenedetto and Herrero [53]. Let us point out that the p-Laplacian equation is invariant under constant u-displacements (i. e., if u is a local weak solution so is $u+c$ for any $c \in \mathbb{R}$). This is a quite convenient property not shared by the porous medium/fast diffusion equation. The equation is also invariant under the symmetry $u \mapsto -u$.

Throughout the chapter we will use the fixed values of the constants

$$p_c = \frac{2n}{n+1}, \qquad r_c = \frac{n(2-p)}{p}, \qquad \vartheta_r = \frac{1}{rp+(p-2)n}. \tag{4.3}$$

Note that $1 < p_c < 2$ for $n > 1$, and $r_c > 1$ for $1 < p < p_c$. See figure in Section 4.10.

Next, we state our main results. By local weak solution we will always refer to the solutions of the fast p-Laplacian equation introduced in Definition 4.1, defined in Q_T, and with $1 < p < 2$. At some places we denote by $|\Omega|$ the Lebesgue volume of a measurable set Ω, typically a ball.

4.2.2 Local Smoothing Effects

Our main result in terms of local upper estimates reads

Theorem 4.1. *Let* u *be a local strong solution of the fast* p*-Laplacian equation with* $1 < p < 2$ *corresponding to an initial datum* $u_0 \in L^r_{\rm loc}(\Omega)$, *where* $\Omega \subseteq \mathbb{R}^n$ *is an open domain containing the ball* $B_R(x_0)$. *If either* $1 < p \le p_c$ *and* $r > r_c$, *or* $p_c < p < 2$ *and* $r \ge 1$, *then there exists two positive constants* C_1 *and* C_2 *such that:*

$$u(x_0,t) \le \frac{C_1}{t^{n\vartheta_r}} \left[\int_{B_R(x_0)} |u_0(x)|^r \,\mathrm{d}x\right]^{p\vartheta_r} + C_2 \left(\frac{t}{R^p}\right)^{\frac{1}{2-p}}. \tag{4.4}$$

Here C_1 *and* C_2 *depend only on* r, p, n; *we recall that* $\vartheta_r > 0$ *under our assumptions.*

Remarks. (i) We point out that a natural choice for R is $R = \mathrm{dist}(x_0, \partial\Omega)$. In this way reference to the inner ball can be avoided. We ask the reader to write the equivalent statement.

(ii) As we have mentioned, using the results of Appendix A2, we deduce that the local strong solutions are in fact locally Hölder continuous.

(iii) This theorem will be a corollary of a slightly more general theorem, namely Theorem 4.8, where the constants C_i depend also on R/R_0. The two terms in the estimates are sharp in a sense that will be explained after the statement of Theorem 4.8.

(iv) Note that changing u into $-u$ and applying the same result we get a bound from below for u. Therefore, we can replace $u(x_0,t)$ by $|u(x_0,t)|$ in the left-hand side of formula (4.4).

(v) The above theorem extends to the limit case $p=1$ with the assumption $r>n$.

(vi) The proof of this Theorem can be extended "as it is" to local strong subsolutions.

Continuous large solutions and extended large solutions. The upper estimate (4.4) will be used to prove the existence of continuous large solutions for the parabolic p-Laplacian equation, cf. Theorem 4.12. Moreover, we prove sharp asymptotic estimates for such large solutions in Theorem 4.13, of the form: $u(x,t)\sim O(\mathrm{dist}(x,\partial\Omega)^{\frac{p}{2-p}}t^{\frac{1}{2-p}})$. See precise expression in (4.51).

4.2.3 Lower bounds for nonnegative solutions

The next results deal with properties of nonnegative solutions. Note that since the equation is invariant under constant u-displacements, the results apply to any local weak solution that is bounded below (and by symmetry $u\mapsto -u$ to any solution that is bounded above). We divide our presentation of the results into several different parts.

A. General positivity estimates. Let u be a nonnegative, continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q=\Omega\times(0,T)$, with $1<p<2$, taking an initial datum $u_0\in L^1_{\mathrm{loc}}(\Omega)$. Let $x_0\in\Omega$ be a fixed point, such that $\mathrm{dist}(x_0,\partial\Omega)>5R$. Consider the *minimal Dirichlet problem*, which is the problem posed in $B_{3R}(x_0)$, with initial data $u_0\chi_{B_R(x_0)}$ and zero boundary conditions. The extinction time $T_m=T_m(u_0,R)$ of the solution of this problem (which is always finite, as results in Subsection 4.7.3 show) is called the *minimal life time*, and indeed it satisfies $T_m(u_0,R)<T(u)$, where $T(u)$ is the (finite or infinite) extinction time of u. In order to pass from the estimate in the center x_0 to the infimum in $B_R(x_0)$, we need that $\mathrm{dist}(x_0,\partial\Omega)>5R$. With all these notations we have:

Theorem 4.2. *Under the previous assumptions, there exists a positive constant $C=C(n,p)$ such that*

$$\inf_{x\in B_R(x_0)} u^{p-1}(x,t)\ge C R^{p-n} t^{\frac{p-1}{2-p}} T_m^{-\frac{1}{2-p}}\int_{B_R(x_0)} u_0(x)\,\mathrm{d}x, \tag{4.5}$$

for any $0<t<t^*$, where $t^*>0$ is a critical time depending on R and on $\|u_0\|_{L^1(B_R)}$, but not on T_m.

The explicit expression the critical time is $t^*=k^*(n,p)R^{p-n(2-p)}\|u_0\|^{2-p}_{L^1(B_R(x_0))}$, cf. (4.86).

The next result is a lower bound for continuous local weak solutions, in the form of *Aronson-Caffarelli estimates*. The main difference with respect to Theorem 4.2 is that this estimate is global in time, and implies the first one.

Theorem 4.3. *Under the assumptions of the last theorem, for any $t\in(0,T_m)$ we have*

$$R^{-n}\int_{B_R(x_0)} u_0(x)\,\mathrm{d}x\le C_1 t^{\frac{1}{2-p}} R^{-\frac{p}{2-p}} + C_2\, t^{-\frac{p-1}{2-p}} T_m^{\frac{1}{2-p}} R^{-p}\inf_{x\in B_R(x_0)} u(x,t)^{p-1}, \tag{4.6}$$

where C_1 and C_2 are positive constants depending only on n and p.

Remark. The presence of T_m may seem awkward since the extinction time is not a direct expression of the data. On the other side, the above estimates hold with the same form in the whole range $1 < p < 2$. We now improve the above estimates by replacing the T_m with some local information on the data, and for this reason it is necessary to separate the results that hold in the supercritical and in the subcritical range.

B. Improved estimates in the "good" fast diffusion range. Let us consider p in the supercritical or "good" fast diffusion range, i. e. $p_c < p < 2$. In this range, we can obtain both lower and upper estimates for T_m in terms of the local L^1 norm of u_0. We prove the following result:

Theorem 4.4. *If $p_c < p < 2$, we have the following upper and lower bounds for the extinction time of the Dirichlet problem T on any ball B_R:*

$$c_1 R^{p-n(2-p)}\|u_0\|^{2-p}_{L^1(B_{R/3})} \le T \le c_2 R^{p-n(2-p)}\|u_0\|^{2-p}_{L^1(B_R)}, \tag{4.7}$$

for some c_1, $c_2 > 0$. Then, the lower estimate (4.5) *reads*

$$\inf_{x\in B_R(x_0)} u(x,t) \ge C(n,p)\left(\frac{t}{R^p}\right)^{\frac{1}{2-p}}, \quad \text{for any } 0 < t < t^*. \tag{4.8}$$

This absolute lower bound is nothing but a lower Harnack inequality, indeed when combined with the upper estimates of Theorem 4.1, it implies the elliptic, forward and even backward inequalities, as in Theorem 4.6, or in [55].

C. Improved estimates in the very fast diffusion range. We now consider $1 < p \le p_c$. In this range the results of the above part B are no longer valid, since an upper estimate of T_m in terms of the L^1 norm of the data is not possible. However, when $u_0 \in L^r_{loc}(\Omega)$ with $r \ge r_c$, we can estimate T_m by $\|u_0\|_{L^{r_c}(B_R)}$, cf. (4.93) or (4.97). In this way we obtain:

Theorem 4.5. *Under the running assumptions, let $1 < p \le p_c$ and let $u_0 \in L^{r_c}(\Omega)$. Let $x_0 \in \Omega$ and $R > 0$ such that $B_{3R}(x_0) \subset \Omega$. Then, the following Aronson-Caffarelli type estimate holds true for any $t \in (0,T)$:*

$$R^{-n}\|u_0\|_{L^1(B_R(x_0))} \le C_1\, t^{\frac{1}{2-p}} R^{-\frac{p}{2-p}} + C_2\,\|u_0\|_{L^{r_c}(B_R(x_0))} R^{-p} t^{-\frac{p-1}{2-p}} \inf_{x\in B_R(x_0)} u^{p-1}(x,t). \tag{4.9}$$

Moreover we have

$$\inf_{B_R(x_0)} u^{p-1}(\cdot,t) \ge C\, R^{p-n} t^{\frac{p-1}{2-p}}\|u_0\|^{-1}_{\mathrm{L}^{r_c}(B_R)}\|u_0\|_{\mathrm{L}^1(B_R)}, \tag{4.10}$$

for any $0 < t < t^*$, with t^* as in Theorem 4.2.

Sharpness of Theorem 4.5. The estimates of Theorem 4.5 are sharp, in the sense that a better estimate in terms of the L^1 norm of u_0 is impossible in the range $1 < p < p_c$. To show this, we produce the following counterexample, imitating a similar one in [34].

Consider first a radially symmetric function $\varphi \in L^1(\mathbb{R}^n)$, with total mass 1 (i. e. $\int \varphi \, dx = 1$), compactly supported and decreasing in $r = |x|$, and rescale it, in order to approximate the Dirac mass δ_0: $\varphi_\lambda(x) = \lambda^n \varphi(\lambda x)$. Let $u(x,t)$ be the solution of the Cauchy problem for the fast p-Laplacian equation with initial data φ, and let $T_1 > 0$ be its finite extinction time. From the scale invariance of the equation, it follows that the solution corresponding to φ_λ is

$$u_\lambda(x,t) = \lambda^n u(\lambda\, x, \lambda^{np-2n+p} t), \quad T_\lambda = T_1 \lambda^{-(np-2n+p)} \to \infty \text{ as } \lambda \to \infty,$$

while the initial data φ_λ has always total mass 1. Hence, estimating T in terms of $\|u_0\|_{L^1}$ is impossible, proving that our estimates are sharp also in the range $1 < p < p_c$.

The limit case $p \to 1$. The positivity result is false in this case, indeed formulas (4.10) and (4.6) degenerate for $p = 1$. Moreover solutions of the 1-Laplacian equation of the form below clearly do not satisfy none of the above positivity estimates. Indeed the function

$$u(x,t) = (1 - \lambda_\Omega t)_+ \chi_\Omega(x), \quad \lambda_\Omega = \frac{P(\Omega)}{|\Omega|}, \quad u_0 = \chi_\Omega$$

is a weak solution to the total variation flow, i.e. the 1-Laplacian, whenever Ω is a set of finite perimeter $P(\Omega)$, satisfying certain condition on the curvature of the boundary, we refer to [3, 19] for further details.

4.2.4 Harnack inequalities

Joining the lower and upper estimates obtained before, we can prove *intrinsic Harnack inequalities* for any $1 < p < 2$. The name *intrinsic* is explained in detail at the beginning of Section 4.8. Let u be a nonnegative, continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q = \Omega \times (0,T)$, with $1 < p < 2$, taking an initial datum $u_0 \in L^r_{\rm loc}(\Omega)$, where $r \ge \max\{1, r_c\}$. Let $x_0 \in \Omega$ be a fixed point, and let $\mathrm{dist}(x_0, \partial\Omega) > 5R$. We have

Theorem 4.6. *Under the above conditions, there exist constants h_1, h_2 depending only on d, p, r, such that, for any $\varepsilon \in [0,1]$ the following inequality holds*

$$\inf_{x \in B_R(x_0)} u(x, t \pm \theta) \ge h_1 \varepsilon^{\frac{rp\vartheta_r}{2-p}} \left[\frac{\|u(t_0)\|_{L^1(B_R)} R^{\frac{n}{r}}}{\|u(t_0)\|_{L^r(B_R)} R^n} \right]^{rp\vartheta_r + \frac{1}{2-p}} u(x_0, t), \tag{4.11}$$

for any

$$t_0 + \varepsilon t^*(t_0) < t \pm \theta < t_0 + t^*(t_0), \quad t^*(t_0) = h_2 R^{p-n(2-p)} \|u(t_0)\|^{2-p}_{L^1(B_R(x_0))}.$$

The proof of this theorem is given in Section 4.8, together with an alternative form that avoids the intrinsic geometry.

Remarks. (i) In the "good fast-diffusion range" $p > p_c$, we can let $r = 1$ and we recover the intrinsic Harnack inequality of [55], that is

$$\inf_{x \in B_R(x_0)} u(x, t \pm \theta) \ge h_1 \varepsilon^{\frac{rp\vartheta_r}{2-p}} u(x_0, t), \qquad \text{for any } t_0 + \varepsilon t^*(t_0) < t \pm \theta < t_0 + t^*(t_0).$$

Let us notice that in this inequality, the ratio of L^r norms simplifies, and the constants h_1, h_2 do not depend on u_0. The size of the intrinsic cylinder is given by t^* as above, in particular we observe that $t^*(t_0) \sim R^{p-n(2-p)}\|u(t_0)\|_{L^1(B_R(x_0))}^{2-p} \sim R^p u(t_0,x_0)^{2-p}$.

(ii) In the subcritical range $p \le p_c$, the Harnack inequality cannot have a universal constant, independent of u_0, cf. [55]. We have thus shown that, if one allows for the constant to depend on u_0, we obtain an intrinsic Harnack inequality, which is a natural continuation of the one in the good range $p > p_c$. The size of the intrinsic cylinders is proportional to a ratio of local L^r norms, but this ratio simplifies only when $p > p_c$.

(iii) We also notice that we need a small waiting time $\varepsilon \in (0,1]$. This waiting time is necessary for the regularization to take place, and thus for the intrinsic inequality to hold, and it can be taken as small as we wish.

(iv) The backward Harnack inequality, i.e., estimate (4.11) taken at time $t-\theta$, is typical of the fast diffusion processes, reflecting an important feature that these processes enjoy, that is extinction in finite time, the solution remaining positive until the finite extinction time. It is easy to see that the backward Harnack inequality does not hold either for the linear heat equation, i.e. $p=2$, or for the degenerate p-Laplacian equation, i.e. $p>2$.

(v) *The Size of Intrinsic Cylinders.* The critical time $t^*(t_0)$ above represents the size of the intrinsic cylinders. In the supercritical fast diffusion range this time can be chosen "a priori" just in terms of the initial datum at $t_0 = 0$, but in the subcritical range its size must change with time; roughly speaking the diffusion is so fast that the local information at t_0 is not relevant after some time, which is represented by $t^*(t_0)$. We must bear in mind that a large class of solutions completely extinguish in finite time.

4.2.5 Special local energy inequality

Theorem 4.7. *Let u be a continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q = \Omega \times (0,T)$, with $1 < p < 2$, in the sense of Definition 4.1, and let $0 \le \varphi \in C_c^2(\Omega)$ be any admissible test function. Then $u_t = \Delta_p u \in L^2_{loc}(Q_T)$ and the following inequality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x + \frac{p}{n}\int_\Omega (\Delta_p u)^2 \varphi \,\mathrm{d}x \le \frac{p}{2}\int_\Omega |\nabla u|^{2(p-1)}\Delta\varphi \,\mathrm{d}x, \tag{4.12}$$

in the sense of distributions in $\mathcal{D}'(0,T)$.

Beyond the interest in itself, Theorem 4.7 has the following consequence that will be important in the sequel:

Corollary 4.1. *Let u be a continuous local weak solution. Then u is a local strong solution in the sense of Definition 4.1.*

We present here a short formal calculation that leads to the inequality (4.12). The complete and rigorous proof of Theorem 4.7 is longer and technical and will be given in Section 4.9.

Formal Proof of Theorem 4.7. We start by differentiating the energy, localized with an admissible test function $\varphi \geq 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x &= p\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla u_t)\varphi \,\mathrm{d}x \\ &= -p\int_\Omega \mathrm{div}(|\nabla u|^{p-2}\nabla u \varphi)\Delta_p u \,\mathrm{d}x \\ &= -p\int_\Omega (\Delta_p u)^2\varphi \,\mathrm{d}x - p\int_\Omega \Delta_p u\, |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x.\end{aligned} \tag{4.13}$$

Next, we estimate the last term in the above calculation. To this end, we use the following formula, (cf. also [39])

$$(\mathrm{div}F)^2 = \mathrm{div}\,(F\mathrm{div}F) - \frac{1}{2}\Delta\left(|F|^2\right) + \mathrm{Tr}\left[\left(\frac{\partial F}{\partial x}\right)^2\right], \tag{4.14}$$

which holds true for any vector field F. We combine it with the following inequality

$$\mathrm{Tr}\left[\left(\frac{\partial F}{\partial x}\right)^2\right] \geq \frac{1}{n}\left[\mathrm{Tr}\left(\frac{\partial F}{\partial x}\right)\right]^2 = \frac{1}{n}\,(\mathrm{div}F)^2\,. \tag{4.15}$$

and we then apply these to the vector field $F = |\nabla u|^{p-2}\nabla u$. We obtain

$$(\Delta_p u)^2 \geq \mathrm{div}\big[|\nabla u|^{p-2}\nabla u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)\big] - \frac{1}{2}\Delta\left(|\nabla u|^{2(p-1)}\right) + \frac{1}{n}(\Delta_p u)^2.$$

We multiply by φ and integrate the above inequality in space, then we plug it into (4.13), and thus get

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p\varphi \,\mathrm{d}x &= -\int_\Omega (\Delta_p u)^2\varphi \,\mathrm{d}x - \int_\Omega (\mathrm{div}F)(F\cdot\nabla\varphi)\,\mathrm{d}x \\ &= -\int_\Omega (\Delta_p u)^2\varphi \,\mathrm{d}x + \int_\Omega \mathrm{div}(F\mathrm{div}F)\varphi \,\mathrm{d}x \\ &\leq -\frac{1}{n}\int_\Omega (\Delta_p u)^2\varphi \,\mathrm{d}x + \frac{1}{2}\int_\Omega \Delta\left(|\nabla u|^{2(p-1)}\right)\varphi \,\mathrm{d}x\end{aligned}$$

where the notation $F = |\nabla u|^{p-2}\nabla u$ is kept for sake of simplicity. A double integration by parts in the last term gives (4.12). Let us finally notice that, in order to perform the integration by parts in the last inequality step above, we need that $\varphi = 0$ and $\nabla\varphi = 0$ on $\partial\Omega$. $\square$

Remarks. (i) The second term in the left-hand side can also be written as $(p/n)\int_\Omega u_t^2\,\varphi\,\mathrm{d}x$ and accounts for local dissipation of the 'energy integral' of the left-hand side. This result continues to hold and it is well known for the linear heat equation, i. e., when $p = 2$.

(ii) Theorem 4.7 may be extended to more general operators, the so-called Φ-Laplacians, under suitable conditions, we refer to Proposition 4.4 and to the remarks at the end of Section 4.9 for such extensions.

(iii) Inequality (4.12) is new and holds also in the limit $p \to 1$ at least formally. In any case, our proof relies on some results concerning regularity that fail when $p = 1$. When $p \to 1$ our inequality reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla u| \varphi \, \mathrm{d}x + \frac{1}{n} \int_\Omega (\Delta_1 u)^2 \varphi \, \mathrm{d}x \le 0,$$

in $\mathcal{D}'(0,T)$, showing in particular that the local energy, in this case the local total variation associated to the 1-Laplacian (or total variation flow) decays in time with some rate. This inequality can be helpful when studying the asymptotic of the total variation flow, a difficult open problem that we do not attack here. A slightly different version of this inequality for $p = 1$ is proven in [3] in the framework of entropy solutions, and is the key tool in proving the L^2_{loc} regularity of the time derivative of entropy solutions.

4.3 Local smoothing effect for bounded strong solutions

We turn now to the proof of Theorem 4.1, that will be divided into two parts: first, we prove it for bounded strong solutions, then (in Section 4.5) we prove the result in the whole generality, for any local strong solution. The result of Theorem 4.1 is obtained as an immediate corollary of the following slightly stronger form of the result.

Theorem 4.8. *Let u be a local strong solution of the fast p-Laplacian equation, with $1 < p < 2$, as in Definition 4.1, corresponding to an initial datum $u_0 \in L^r_{\mathrm{loc}}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is any open domain containing the ball $B_{R_0}(x_0)$. If either $1 < p \le p_c$ and $r > r_c$ or $p_c < p < 2$ and $r \ge 1$, then there exist two positive constants C_1 and C_2 such that for any $0 < R < R_0$ we have:*

$$\sup_{(x,\tau)\in B_R\times(\tau_0 t, t]} u(x,\tau) \le \frac{C_1}{t^{n\vartheta_r}} \left[\int_{B_{R_0}} |u_0(x)|^r \, \mathrm{d}x\right]^{p\vartheta_r} + C_2 \left(\frac{t}{R_0^p}\right)^{\frac{1}{2-p}}, \tag{4.16}$$

where $\tau_0 = [(R_0 - R)/(R_0 + R)]^p$ and

$$C_1 = K_1 \left[\frac{R_0 + R}{R_0 - R}\right]^{p(n+p)\vartheta_r}, \quad C_2 = K_2 \left[\frac{R_0 + R}{R_0 - R}\right]^{p(n+p)\vartheta_r} \left[K_3 \left(\frac{R_0 - R}{R_0 + R}\right)^{\frac{2-p-rp}{2-p}} + K_4\right]^{p\vartheta_r},$$

with K_i, $i = 1, 2, 3$, depending only on r, p, n, and $K_4 = \omega_n$ if $r > 1$, $K_4 = \omega_n + L(n,p) > \omega_n$ if $r = 1$. ω_n is the measure of the unit ball of $\mathbb{R}^n$, and we recall that $\vartheta_r = 1/[rp+(p-2)n] > 0$.

Note that Theorem 4.1 follows immediately from Theorem 4.8 by letting $R = 0$ and considering, for $x_0 \in \Omega$ fixed, the ball centered in x_0 and tangent to $\partial\Omega$.

Interpreting the two terms in the estimate. The right-hand side of (4.16) is the sum of two independent terms. Let us discuss them separately.

(i) The first term concentrates the influence of the initial data u_0. It has the exact form of the global smoothing effect (i. e. the smoothing effect for solutions defined in the whole space with initial data in $L^r(\mathbb{R}^n)$ or in the Marcinkiewicz space $\mathcal{M}^r(\mathbb{R}^n)$), cf. Theorem 11.4

of [127]. Hence, if we pass to the limit in (4.16) as $R_0 \to \infty$, we recover the global smoothing effect on $\mathbb{R}^n$ (however, the constant need not to be optimal).

(ii) The second term appears as a correction term when passing from global estimates to local upper bounds. It can be interpreted as an *absolute damping* of all external influences due to the form of the diffusion operator, more precisely, due to fast diffusion. Let us note that, by shrinking the ball B_{R_0} (and at the same time the smaller ball B_R), the influence of this term increases, while that of the first one tends to disappear.

A remarkable consequence of this absolute damping is the existence of large solutions that we will discuss in Section 4.4. Indeed, there is an explicit large solution with zero initial data that has precisely the form of the last term in (4.16) with $R = 0$ – or in the corresponding term in (4.4) – which means that such term has an optimal form that cannot be improved without information on the boundary data (again, the constant need not to be optimal).

We first prove Theorem 4.8 for bounded local strong solutions, then we will remove the assumption of local boundedness in Section 4.5. The proof of Theorem 4.8 for bounded local strong solutions consists of combining L^r_{loc}-stability estimates, together with a space-time local smoothing effect, proved via Moser-style iteration. This will be the subject of the next subsections.

4.3.1 Space-time local smoothing effects

In this section we prove a form of the Local Smoothing Effect for the p-Laplacian equation, with $1 < p < 2$. More precisely, we are going to prove that L^r_{loc} regularity in space-time for some $r \geq 1$ implies L^∞_{loc} estimates in space-time.

Theorem 4.9. *Let u be a bounded local strong solution of the p-Laplacian equation, $1 < p < 2$, and let either $1 < p \leq p_c$ and $r > r_c$ or $p_c < p < 2$ and $r \geq 1$. Then, for any two parabolic cylinders $Q_1 \subset Q$, where $Q = B_{R_0} \times (T_0, T]$ and $Q_1 = B_R \times (T_1, T]$, with $0 < R < R_0$ and $0 \leq T_0 < T_1 < T$, we have:*

$$\sup_{Q_1} |u| \leq K \left[\frac{1}{(R_0 - R)^p} + \frac{1}{T_1 - T_0} \right]^{\frac{p+n}{rp+n(p-2)}} \left(\iint_Q u^r \, \mathrm{d}x \, \mathrm{d}t + |Q| \right)^{\frac{p}{rp+n(p-2)}}, \tag{4.17}$$

where $K > 0$ is a constant depending only on r, p and n.

Remarks: (i) Under the assumptions of Theorem 4.9, the local boundedness in terms of some space-time integrals of the solution u is proved as Theorem 3.8 in [57]. In this section, we only give a slight, quantitative improvement of it, which in fact appears in this form in [55], but only for the"good" range $p_c < p < 2$ and for L^1_{loc} initial data. We prove it here for all $1 < p < 2$.

The proof of our main local upper bound, in the form of Theorems (4.4) or (4.16) is given in Subsection 4.3.3 , in which we combine the above time-space smoothing effect (4.16) with the L^r_{loc} stability estimates (4.33) of Subsection 4.3.2.

(ii) This space-time Smoothing Effect holds also for the equation with bounded variable coefficients, as well as for more general operators such as Φ-Laplacians or as in the general framework treated in [55]. We are not addressing this generality since the rest of the theory is not immediate.

We divide the proof into several steps, following the same general program used by two of the authors in [34] for the fast diffusion equation.

Step 1. A space-time energy inequality

Let us consider a bounded local strong solution u defined in a parabolic cylinder $Q = B_{R_0} \times (T_0, T]$. Take $R < R_0$, $T_1 \in (T_0, T]$ and consider a smaller cylinder $Q_1 = B_R \times (T_1, T] \subset Q$. Under these assumptions, we prove:

Lemma 4.1. *For every $1 < p < 2$ and $r > 1$, the following inequality holds:*

$$\begin{aligned}\int_{B_R} u^r(x,T)\,\mathrm{d}x + \iint_{Q_1} \left|\nabla u^{\frac{p+r-2}{p}}\right|^p \,\mathrm{d}x\,\mathrm{d}t \le C(p,r,n)\left[\frac{1}{(R_0-R)^p} + \frac{1}{T_1-T_0}\right] \\ \times \left[\iint_Q (u^r + u^{r+p-2})\,\mathrm{d}x\,\mathrm{d}t\right].\end{aligned} \tag{4.18}$$

The same holds also for local subsolutions in the sense of Definition 4.1.

Proof. (i) We multiply first the p-Laplacian equation by $u^{r-1}\varphi^p$, where $\varphi = \varphi(x,t)$ is a smooth test function with compact support. Integrating in Q we obtain:

$$\begin{aligned}\iint_Q u^{r-1}u_t\varphi^p\,\mathrm{d}x\,\mathrm{d}t &= \iint_Q u^{r-1}\Delta_p u\varphi^p\,\mathrm{d}x\,\mathrm{d}t = -\iint_Q |\nabla u|^{p-2}\nabla u\cdot\nabla(u^{r-1}\varphi^p)\,\mathrm{d}x\,\mathrm{d}t \\ &= -(r-1)\iint_Q |\nabla u|^p u^{r-2}\varphi^p\,\mathrm{d}x\,\mathrm{d}t - p\iint_Q u^{r-1}|\nabla u|^{p-2}\nabla u\cdot\varphi^{p-1}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &= -(r-1)\iint_Q \left|u^{\frac{r-2}{p}}\nabla u\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t - p\iint_Q u^{r-1}|\nabla u|^{p-2}\nabla u\cdot\varphi^{p-1}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

hence

$$\iint_Q \left[u^{r-1}u_t + \frac{(r-1)p^p}{(r+p-2)^p}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\right]\varphi^p\,\mathrm{d}x\,\mathrm{d}t = -p\iint_Q |\nabla u|^{p-2}u^{r-1}\varphi^{p-1}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{4.19}$$

In order to estimate the term in the right-hand side we use the inequality $\overrightarrow{a}\cdot\overrightarrow{b} \le \frac{|\overrightarrow{a}|^\sigma}{\varepsilon\sigma} + \frac{|\overrightarrow{b}|^\gamma}{\gamma}\varepsilon^{\frac{\gamma}{\sigma}}$, which holds for any vectors $\overrightarrow{a}$, $\overrightarrow{b}$, for any $\varepsilon > 0$ and for any exponents σ and γ with $\sigma^{-1} + \gamma^{-1} = 1$. The choice of vectors and exponents as below

$$\overrightarrow{a} = u^{\frac{r+p-2}{p}}\nabla\varphi,\ \sigma = p/(p-1),\ \overrightarrow{b} = u^{\frac{(r-2)(p-1)}{p}}\varphi^{p-1}\,|\nabla u|^{p-2}\nabla u,\ \gamma = p\,,$$

lead to

$$\begin{aligned}-p\iint_Q |\nabla u|^{p-2}u^{r-1}\varphi^{p-1}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \le \frac{(p-1)p^p}{(r+p-2)^p}\varepsilon^{\frac{1}{p-1}}\iint_Q \left|\nabla u^{\frac{r+p-2}{p}}\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{1}{\varepsilon}\iint_Q u^{r+p-2}|\nabla\varphi|^p\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

On the other hand, we integrate the first term by parts with respect to the time variable:

$$\begin{aligned}\frac{1}{r}\iint_Q \partial_t(u^r)\varphi^p \,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{r}\int_0^T\int_{B_{R_0}} \partial_t(u^r)\varphi^p\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{r}\int_{B_{R_0}}\Big[u(x,T)^r\varphi(x,T)^p - u(x,0)^r\varphi(x,0)^p\Big]\,\mathrm{d}x - \frac{p}{r}\iint_Q u^r\varphi^{p-1}\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Joining equality (4.19) and the previous estimates, and choosing

$$\varepsilon = \left(\frac{r-1}{r+p-2}\right)^{p-1},$$

we obtain:

$$\begin{aligned}&\frac{1}{r}\int_{B_{R_0}}\Big[u(x,T)^r\varphi(x,T)^p - u(x,0)^r\varphi(x,0)^p\Big]\,\mathrm{d}x - \frac{p}{r}\iint_Q u^r\varphi^{p-1}\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &+\frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\iint_Q\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t \le \left(\frac{r+p-2}{r-1}\right)^{p-1}\iint_Q u^{r+p-2}|\nabla\varphi|^p\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

(ii) We now impose some additional conditions on φ, namely we assume that $0\le\varphi\le 1$ in Q, $\varphi\equiv 0$ outside Q and $\varphi\equiv 1$ in $\overline{Q_1}$. Moreover, we ask φ to satisfy:

$$|\nabla\varphi|\le\frac{C(\varphi)}{R_0-R},\qquad |\partial_t\varphi|\le\frac{C(\varphi)^p}{T_1-T_0}$$

in the annulus $B_{R_0}\setminus B_R$, and $\varphi(x,0)=0$ for any $x\in B_R$. With these notations, we can continue the previous estimates as follows:

$$\begin{aligned}&\frac{1}{r}\int_{B_{R_0}} u(x,T)^r\varphi(x,T)^p\,\mathrm{d}x + \frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\iint_Q\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{p}{r}\iint_Q u^r\varphi^{p-1}|\partial_t\varphi|\,\mathrm{d}x\,\mathrm{d}t + \left(\frac{r+p-2}{r-1}\right)^{p-1}\iint_Q u^{r+p-2}|\nabla\varphi|^p\,\mathrm{d}x\,\mathrm{d}t\\ &\le C^p\left[\frac{p}{r}\frac{1}{T_1-T_0}+\left(\frac{r+p-2}{r-1}\right)^{p-1}\frac{1}{(R_0-R)^p}\right]\iint_Q\left(u^r+u^{r+p-2}\right)\,\mathrm{d}x\,\mathrm{d}t\\ &\le 2C^p\max\left\{\frac{p}{r},\left(\frac{r+p-2}{r-1}\right)^{p-1}\right\}\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\iint_Q\left(u^r+u^{r+p-2}\right)\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

We now estimate the left-hand side of the last inequality, in view of the properties of φ:

$$\begin{aligned}&\frac{1}{r}\int_{B_{R_0}} u(x,T)^r\varphi(x,T)^p\,\mathrm{d}x + \frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\iint_Q\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\varphi^p\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\ge\frac{1}{r}\int_{B_R}u^r(x,T)\,\mathrm{d}x + \frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\iint_{Q_1}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\ge\min\left\{\frac{1}{r},\frac{(r-1)^2p^p}{(r+p-2)^{p+1}}\right\}\left(\int_{B_R}u(x,T)^r\,\mathrm{d}x + \iint_{Q_1}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}$$

Joining all the previous calculations, we arrive to the conclusion. The same proof can be repeated also for local subsolutions as in Definition 4.1. $\square$

Remark: A closer inspection of the above proof allows us to evaluate the constant $C(r,p)$ in a more precise way. Indeed, we observe that

$$C(r,p) = 2C(\varphi)\max\left\{\frac{p}{r}, \left(\frac{r+p-2}{r-1}\right)^{p-1}\right\}\min\left\{\frac{1}{r}, \frac{(r-1)^2 p^p}{(r+p-2)^{p+1}}\right\}^{-1}.$$

By evaluating the dependence in r of the constants, we remark that, for r sufficiently large, $C(r,p) = O(r)$. Hence, $C(r,p)$ is bounded from below by a constant independent of r, but from above it is not. In any case, as we will see, the rate $C(r,p) = O(r)$ is good for our aims. We will use the space-time energy inequality in the following improved version.

Lemma 4.2. *Under the running assumptions, we have:*

$$\begin{aligned}\sup_{s\in(T_1,T)}\int_{B_R} u^r(x,s)\,\mathrm{d}x + \int_{T_1}^{T}\int_{B_R}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p \\ \le C(r,p)\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\int_{T_0}^{T}\int_{B_{R_0}}\left(u^{r+p-2}+u^r\right)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.20}$$

Moreover, if u is a weak subsolution and $u \ge 1$, we have:

$$\begin{aligned}\sup_{s\in(T_1,T)}\int_{B_R} u^r(x,s)\,\mathrm{d}x + \int_{T_1}^{T}\int_{B_R}\left|\nabla u^{\frac{r+p-2}{p}}\right|^p\,\mathrm{d}x\,\mathrm{d}t \\ \le C(r,p)\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\int_{T_0}^{T}\int_{B_{R_0}} u^r\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.21}$$

Proof. We recall a well known property of the supremum, namely there exists $t_0 \in (T_1,T)$ such that

$$\frac{1}{2}\sup_{s\in(T_1,T)}\int_{B_R} u^r(x,s)\,\mathrm{d}x \le \int_{B_R} u^r(x,t_0)\,\mathrm{d}x.$$

Since $T_0 \le T_1 < t_0 < T$, we can apply Lemma 4.1 and obtain

$$\int_{B_R} u^r(x,t_0)\,\mathrm{d}x \le C(r,p)\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\int_{T_0}^{t_0}\int_{B_{R_0}}\left(u^{r+p-2}+u^r\right)\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, also the second term can be estimated by Lemma 4.1:

$$\int_{T_1}^{T}\int_{B_R}|\nabla u^{\frac{r+p-2}{p}}|^p\,\mathrm{d}x\,\mathrm{d}t \le C(r,p)\left[\frac{1}{T_1-T_0}+\frac{1}{(R_0-R)^p}\right]\int_{T_0}^{T}\int_{B_{R_0}}(u^{r+p-2}+u^r)\,\mathrm{d}x\,\mathrm{d}t.$$

Summing up the two previous inequalities, we obtain inequality (4.20). If $u \ge 1$, is a subsolution, then $u^{r+p-2} \le u^r$, hence we immediately get (4.21). $\square$

Step 2: An iterative form of the Sobolev inequality

We state the classical Sobolev inequality in a different form, adapted for the Moser-type iteration.

Lemma 4.3. *Let $f \in L^p(Q)$ with $\nabla f \in L^p(Q)$. Then, for any $\sigma \in (1, \sigma^*)$, for any $0 \le T_0 < T_1$ and $R > 0$, the following inequality holds:*

$$\begin{aligned} \int_{T_0}^{T_1} \int_{B_R} f^{p\sigma} \, \mathrm{d}x \, \mathrm{d}t \le 2^{p-1} \mathcal{S}_p^p & \left[\int_{T_0}^{T_1} \int_{B_R} \left(f^p + R^p \left| \nabla f \right|^p \right) \, \mathrm{d}x \, \mathrm{d}t \right] \\ & \times \sup_{t \in (T_0, T_1)} \left[\frac{1}{R^n} \int_{B_R} f^{p(\sigma-1)q}(t, x) \, \mathrm{d}x \right]^{\frac{1}{q}}, \end{aligned} \tag{4.22}$$

where

$$p^* = \frac{np}{n-p}, \qquad \sigma^* = \frac{p^*}{p} = \frac{n}{n-p}, \qquad q = \frac{p^*}{p^* - p} = \frac{n}{p},$$

and the constant $\mathcal{S}_p$ is the constant of the classical Sobolev inequality.

Proof. We first prove the inequality for $R = 1$. We write:

$$\int_{B_1} f^{p\sigma} \, \mathrm{d}y = \int_{B_1} f^p f^{p(\sigma-1)} \, \mathrm{d}y \le \left(\int_{B_1} f^{p^*} \, \mathrm{d}y \right)^{\frac{p}{p^*}} \left(\int_{B_1} f^{p(\sigma-1)q} \, \mathrm{d}y \right)^{\frac{1}{q}},$$

We use now the standard Sobolev inequality in the first factor of the right-hand side:

$$\|f\|_{p^*}^p \le \mathcal{S}_p^p (\|f\|_p + \|\nabla f\|_p)^p \le 2^{p-1} \mathcal{S}_p^p (\|f\|_p^p + \|\nabla f\|_p^p).$$

Passing to the supremum in time in the second factor of the right-hand side, then integrating the inequality in time, over (T_0, T_1), we obtain the desired form for $R = 1$. Finally, the change of variable $x = Ry$ allow to obtain (4.22) for any $R > 0$. □

Step 3: Preparation of the iteration.

Let us first define $v(x, t) = \max\{u(x, t), 1\}$. We remark that v is a local weak subsolution of the p-Laplacian equation in the sense of Definition 4.1. Moreover $u \le v \le 1 + v$ for any $(x, t) \in Q$.

We now let $f^p = v^{r+p-2}$ in the iterative Sobolev inequality (4.22) and we apply it for $Q_1 \subset Q$ as in the statement of Theorem 4.9. We then obtain:

$$\begin{aligned} \int_{T_1}^{T} \int_{B_R} v^{\sigma(r+p-2)} \, \mathrm{d}x \, \mathrm{d}t \le 2^{p-1} \mathcal{S}_p^p & \left[\iint_Q \left(v^{r+p-2} + R^p \left| \nabla v^{\frac{r+p-2}{p}} \right|^p \right) \, \mathrm{d}x \, \mathrm{d}t \right] \\ & \times \left[\sup_{t \in [T_1, T]} \frac{1}{R^n} \int_{B_R} v^{(r+p-2)(\sigma-1)q} \, \mathrm{d}x \right]^{\frac{1}{q}}. \end{aligned} \tag{4.23}$$

Since $v \geq 1$, we can use the space-time energy inequality (4.21) to estimate both terms in the right-hand side:

$$\begin{aligned}\iint_Q \left(v^{r+p-2} + R^p \left|\nabla v^{\frac{r+p-2}{p}}\right|^p\right) \, dx \, dt \\ &\leq R^p C(r,p) \left[\frac{1}{(R_0-R)^p} + \frac{1}{T_1 - T_0}\right] \left[\iint_Q v^r \, dx \, dt\right] + \iint_Q v^r \, dx \, dt \\ &\leq 2R^p C(r,p) \left[\frac{1}{(R_0-R)^p} + \frac{1}{T_1 - T_0}\right] \left[\iint_Q v^r \, dx \, dt\right].\end{aligned} \tag{4.24}$$

As for the other term, we use again the space-time energy inequality (4.21), but we replace $r > 1$ by $(r+p-2)(\sigma-1)q > 1$, and we get

$$\begin{aligned}\sup_{t\in[T_1,T]} \frac{1}{R^n} \int_{B_R} v^{(r+p-2)(\sigma-1)q} \, dx &\leq \frac{C(r,p)}{R^n} \left[\frac{1}{(R_0-R)^p} + \frac{1}{T_1 - T_0}\right] \\ &\quad \times \left[\iint_{Q_0} v^{(r+p-2)(\sigma-1)q} \, dx \, dt\right].\end{aligned} \tag{4.25}$$

Plugging (4.24) and (4.25) into (4.23), we obtain

$$\begin{aligned}\iint_Q v^{\sigma(r+p-2)} \, dx \, dt &\leq 2^{p-1} \mathcal{S}_p^p C(r,p)^{1+\frac{1}{q}} \left[\frac{1}{(R_0-R)^p} + \frac{1}{T_1 - T_0}\right]^{1+\frac{1}{q}} \\ &\quad \times \left[\iint_{Q_0} v^r \, dx \, dt\right] \left[\iint_{Q_0} v^{(\sigma-1)(r+p-2)q} \, dx \, dt\right]^{\frac{1}{q}}.\end{aligned} \tag{4.26}$$

notice that R cancels, since $R^{p-n/q} = 1$. $\square$

Step 4. Choosing the exponents

We begin by choosing $r = q(r+p-2)(\sigma-1) := r_0$, with $\sigma \in (1, \sigma^*)$. This implies that $\sigma = 1 + rp/n(r+p-2)$. This is always larger than 1, but it has to satisfy $\sigma < \sigma^* = n/(n-p)$, hence we need that $r > n(2-p)/p := r_c$. We remark that $r_c > 1$ if and only if $p < p_c$. We define next

$$r_1 = (r_0 + p - 2)\sigma = \left(1 + \frac{1}{q}\right) r_0 + p - 2,$$

and we see that $r_1 > r_0$ if and only if $r_0 > r_c$. In a natural way, we iterate this construction:

$$r_{k+1} = r_k \left(1 + \frac{1}{q}\right) + p - 2,$$

and we note that $r_{k+1} > r_k$ if and only if $r_k > r_0 > r_c$. Moreover, we can provide an explicit formula for the exponents

$$\begin{aligned}r_{k+1} &= r_k(1 + \frac{1}{q}) + p - 2 = \left(1 + \frac{1}{q}\right)^{k+1} r_0 + (p-2) \sum_{j=0}^{k} \left(1 + \frac{1}{q}\right)^j \\ &= \left(1 + \frac{1}{q}\right)^{k+1} \left[r_0 + (p-2) \sum_{l=1}^{k+1} \left(1 + \frac{1}{q}\right)^{-l}\right] = \left(1 + \frac{1}{q}\right)^{k+1} \left[r_0 - (2-p)q\right] + q(2-p).\end{aligned} \tag{4.27}$$

Let us calculate two useful limits of the exponents:

$$\lim_{k\to\infty}\frac{\left(1+\frac{1}{q}\right)^{k+1}}{r_{k+1}}=\frac{1}{r_0+(p-2)q},\quad \lim_{k\to\infty}\frac{1}{r_{k+1}}\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^{j}=\frac{q}{r_0+(p-2)q}. \tag{4.28}$$

We are now ready to rule the iteration process.

Step 5. The iteration

The iteration process consists in writing the inequality (4.26) with the exponents introduced in the previous step. The first step then reads

$$\begin{aligned}\left[\iint_Q v^{r_1}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_1}} &\le \left\{2^{p-1}\mathcal{S}_p^p C(r_0,p)^{1+\frac{1}{q}}\left[\frac{1}{(R_0-R)^p}+\frac{1}{T_1-T_0}\right]^{1+\frac{1}{q}}\right\}^{\frac{1}{r_1}}\\ &\quad\times\left[\iint_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\left(1+\frac{1}{q}\right)\frac{1}{r_1}}=I_{0,1}^{\frac{1}{r_1}}\left[\iint_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\left(1+\frac{1}{q}\right)\frac{1}{r_1}}.\end{aligned} \tag{4.29}$$

As for the general iteration step, we have to construct a sequence of cylinders Q_k such that $Q_{k+1}\subset Q_k$, with the convention $Q_1=Q$, and apply it to inequality (4.26). We let $Q_k=B_{R_k}\times(T_k,T]$, with $R_{k+1}<R_k$ and $T_k<T_{k+1}<T$. The k-th step then reads

$$\left[\iint_{Q_{k+1}} v^{r_{k+1}}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_{k+1}}}\le I_{k,k+1}^{\frac{1}{r_{k+1}}}\left[\iint_{Q_k} v^{r_k}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_k}\left(1+\frac{1}{q}\right)\frac{r_k}{r_{k+1}}}, \tag{4.30}$$

where

$$I_{k,k+1}:=2^{p-1}\mathcal{S}_p^p C(r_k,p)^{1+\frac{1}{q}}\left[\frac{1}{(R_k-R_{k+1})^p}+\frac{1}{T_{k+1}-T_k}\right]^{1+\frac{1}{q}}.$$

Iterating now (4.30) we obtain

$$\left[\iint_{Q_{k+1}} v^{r_{k+1}}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{r_{k+1}}}\le I_{k,k+1}^{\frac{1}{r_{k+1}}}I_{k-1,k}^{\left(1+\frac{1}{q}\right)\frac{1}{r_{k+1}}}\ldots I_{0,1}^{\left(1+\frac{1}{q}\right)^k\frac{1}{r_{k+1}}}\left[\iint_{Q_0} v^{r_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\left(1+\frac{1}{q}\right)^{k+1}\frac{1}{r_{k+1}}}. \tag{4.31}$$

In order to get uniform estimates for $I_{k,k+1}$, we have to impose some further conditions on R_k and T_k. More precisely, we choose a decreasing sequence $R_k\to R_\infty>0$ such that $R_k-R_{k+1}=\rho/k^2$ and an increasing sequence of times $T_k\to T_\infty<T$ such that $T_{k+1}-T_k=\tau/k^{2p}$. Moreover, we see that

$$\tau=\frac{T_\infty-T_0}{\sum_k\frac{1}{k^{2p}}}>0,\qquad \rho=\frac{R_0-R_\infty}{\sum_k\frac{1}{k^2}}>0.$$

Recall also that the constants $C(r_j,p)\le C_0(p)r_j$, so that

$$I_{j,j+1}\le 2^{p-1}\mathcal{S}_p^p C(p)\left[j^{2p}r_j\left(\frac{1}{\tau}+\frac{1}{\rho^p}\right)\right]^{1+\frac{1}{q}}\le J_0J_1^{1+\frac{1}{q}}\left(j^{2p}r_j\right)^{1+\frac{1}{q}},$$

where $J_0 = 2^{p-1}\mathcal{S}_p^p C(p)$, $J_1 = \tau^{-1} + \rho^{-p}$ are constants that do not depend on r. Hence, we obtain:

$$I_{k,k+1}^{\frac{1}{r_{k+1}}} I_{k-1,k}^{\left(1+\frac{1}{q}\right)\frac{1}{r_{k+1}}} \dots I_{0,1}^{\left(1+\frac{1}{q}\right)^k \frac{1}{r_{k+1}}} \le \left[J_0 J_1^{1+\frac{1}{q}}\right]^{\frac{1}{r_{k+1}}\sum_{j=0}^{k}\left(1+\frac{1}{q}\right)^j} \prod_{j=0}^{k} \left(j^{2p} r_j\right)^{\frac{1}{r_{k+1}}\left(1+\frac{1}{q}\right)^{k+1-j}}$$

and it remains to study the convergence of the products in the right-hand side. To this end, we take logarithms and we write:

$$\begin{aligned} \log\left[\prod_{j=0}^{k} (j^{2p} r_j)^{\frac{1}{r_{k+1}}\left(1+\frac{1}{q}\right)^{k+1-j}}\right] &= \sum_{j=0}^{k} \frac{1}{r_{k+1}}\left(1+\frac{1}{q}\right)^{k+1-j} \big(\log(r_j) + 2p\log j\,\big) \\ &= \frac{1}{r_{k+1}}\left(1+\frac{1}{q}\right)^{k+1} \sum_{j=0}^{k}\left[\frac{\log(r_j)}{\left(1+\frac{1}{q}\right)^j} + 2p\frac{\log j}{\left(1+\frac{1}{q}\right)^j}\right]. \end{aligned}$$

It is immediate to check that the series obtained in the right-hand side are convergent. We can pass to the limit as $k \to \infty$ in (4.31) taking into account of (4.28)

$$\sup_{Q_\infty} |v| \le J_0^{\frac{q}{r_0+(p-2)q}} J_1^{\frac{q+1}{r_0+(p-2)q}} C(n,p) \left[\iint_{Q_0} v^{r_0}\, \mathrm{d}x\, \mathrm{d}t\right]^{\frac{1}{r_0+(p-2)q}}. \tag{4.32}$$

This last estimate holds true for cylinders $Q_\infty \subset Q_0$ and it blows-up as $Q_\infty \to Q_0$, since the constant $J_1 = \tau^{-1} + \rho^{-p}$ blows-up in such a limit. Once we let $Q_\infty = B_{R_\infty} \times (T_\infty, T]$, we see that Q_∞ has to be strictly contained in the initial cylinder $Q_0 = Q$. We finally rewrite the estimate (4.32) in terms of T_∞ and R_∞, in the following way

$$\sup_{Q_\infty} |v| \le C(r_0,p,n)\left[\frac{1}{(R_0 - R_\infty)^p} + \frac{1}{T_\infty - T_0}\right]^{\frac{q+1}{r_0+(p-2)q}} \left[\iint_{Q_0} v^{r_0}\, \mathrm{d}x\, \mathrm{d}t\right]^{\frac{1}{r_0+(p-2)q}}. \quad \square$$

Step 6. End of the proof of Theorem 4.9

The result of Theorem 4.9 is given in terms of the local strong solution u. We then recall that $u \le v \le 1 + u$, hence

$$\begin{aligned} \sup_{Q_\infty} |u| \le \sup_{Q_\infty} |v| &\le C(r_0,p,n)\left[\frac{1}{(R_0 - R_\infty)^p} + \frac{1}{T_\infty - T_0}\right]^{\frac{q+1}{r_0+(p-2)q}} \left[\iint_{Q_0} v^{r_0}\, \mathrm{d}x\, \mathrm{d}t\right]^{\frac{1}{r_0+(p-2)q}} \\ &\le C(r_0,p,n)\left[\frac{1}{(R_0 - R_\infty)^p} + \frac{1}{T_\infty - T_0}\right]^{\frac{q+1}{r_0+(p-2)q}} \left[\iint_{Q_0} u^{r_0}\, \mathrm{d}x\, \mathrm{d}t + |Q_0|\right]^{\frac{1}{r_0+(p-2)q}}. \end{aligned}$$

The proof is concluded once we go back to the original notations as in the statement of Theorem 4.9, namely we let $r = r_0$, $R_\infty = R < R_0$, $T_\infty = T_1 \in (T_0, T)$ and $q = n/p$. $\square$

4.3.2 Behaviour of local L^r-norms. L^r-stability

In this subsection we state and prove an L^r_{loc}-stability results, namely we compare local L^r norms at different times.

Theorem 4.10. *Let $u \in C\big((0,T) : W^{1,p}_{loc}(\Omega)\big)$ be a bounded local strong solution of the fast p-Laplacian equation, with $1 < p < 2$. Then, for any $r > 1$ and any $0 < R < R_0 \le \operatorname{dist}(x_0, \partial\Omega)$ we have the following upper bound for the local L^r norm:*

$$\left[\int_{B_R(x_0)} |u|^r(x,t)\,\mathrm{d}x\right]^{\frac{2-p}{r}} \le \left[\int_{B_{R_0}(x_0)} |u|^r(x,s)\,\mathrm{d}x\right]^{\frac{2-p}{r}} + C_r\,(t-s), \tag{4.33}$$

for any $0 \le s \le t \le T$, where

$$C_r = \frac{C_0}{(R_0-R)^p}|B_{R_0} \setminus B_R|^{\frac{2-p}{r}}, \qquad \text{if } r > 1, \tag{4.34}$$

with C_1 and C_0 depending on p and on the dimension n. Moreover, C_0 depends also on r and blows up when $r \to +\infty$.

Remarks: (i) Theorem 4.10 implies that, whenever $u(\cdot,s) \in L^r_{loc}(\Omega)$, for some time $s \ge 0$ and some $r \ge 1$, then $u(\cdot,t) \in L^r_{loc}(\Omega)$, for all $t > s$, and there is a quantitative estimate of the evolution of the L^r_{loc}-norm. This is what we call L^r_{loc}-stability.
(ii) We remark that the result of Theorem 4.10 is false for $p \ge 2$, since any L^r_{loc} stability result necessarily involves the control of the boundary data; on the other hand, this local upper bound may be extended also to the limit case $p \to 1$.
(iii) Let us examine the behavior of the constant C_r. We see that it blows-up as $R \to R_0$. Indeed, we can write in that limit:

$$C_r(R, R_0, p, n) \sim (R_0 - R)^{\frac{2-p-rp}{r}},$$

and in our conditions $2 - p - rp < 0$. On the other hand, if we choose proportional radii, say $R = R_0/2$, we get

$$C_r = C(n,p,r) R_0^{-(r-r_c)p/r}.$$

In the limit $R_0 \to \infty$, we recover the standard monotonicity of the global $L^r(\mathbb{R}^n)$-norms, when $r > r_c$.
(iv) Theorem 4.10 holds true also for more general nonlinear operators, the so-called Φ-Laplacians, or for operators with variable coefficients satisfying the standard structure conditions of [52], recalled in Section 8. The proof is similar and we leave it to the interested reader.

Proof. Let us calculate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J(u)\varphi\,\mathrm{d}x &= \int_\Omega |J'(u)|\Delta_p(u)\varphi\,\mathrm{d}x = -\int_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla\left(J'(u)\varphi\right)\,\mathrm{d}x \\
&= -\int_\Omega |\nabla u|^p\,J''(u)\varphi\,\mathrm{d}x - \int_\Omega |\nabla u|^{p-2}\,|J'(u)|\nabla\varphi\cdot\nabla u\,\mathrm{d}x \\
&\le -\int_\Omega |\nabla u|^p\,J''(u)\varphi\,\mathrm{d}x + \int_\Omega |\nabla u|^{p-1}\,|J'(u)|\,|\nabla\varphi|\,\mathrm{d}x,
\end{aligned} \tag{4.35}$$

where J is a suitable convex function that will be explicitly chosen afterwards. All the integration by parts are justified in view of the Hölder regularity of the solution and by Corollary 4.1. We now estimate the last integral

$$
\begin{aligned}
I_1 = \int_\Omega |\nabla u|^{p-1}\, |J'(u)|\, |\nabla\varphi| \,\mathrm{d}x &\le \left[\int_\Omega |\nabla u|^p\, J''(u)\varphi\right]^{\frac{p-1}{p}} \left[\int_\Omega \frac{|J'(u)|^p}{[J''(u)]^{p-1}} \frac{|\nabla\varphi|^p}{\varphi^{p-1}} \,\mathrm{d}x\right]^{\frac{1}{p}} \\
&\le \underbrace{\left[\int_\Omega |\nabla u|^p\, J''(u)\varphi\right]^{\frac{p-1}{p}}}_{a^{\frac{p-1}{p}}} \underbrace{\left[\int_\Omega \frac{|J'(u)|^{p\delta'}}{[J''(u)]^{(p-1)\delta'}} \varphi \,\mathrm{d}x\right]^{\frac{1}{p\delta'}} \left[\int_\Omega \frac{|\nabla\varphi|^{p\delta}}{\varphi^\gamma} \,\mathrm{d}x\right]^{\frac{1}{p\delta}}}_{b},
\end{aligned}
\tag{4.36}
$$

where $\gamma = \delta(p-1+1/\delta') = p\delta - 1$. In the first line we have used Hölder's inequality with conjugate exponents $p/(p-1)$ and p. In the second line, Hölder's inequality with conjugate exponents $\delta > 1$ and $\delta' = \delta/(\delta-1)$. We now use the numerical inequality

$$
a^{\frac{p-1}{p}} b \le \frac{p-1}{\varepsilon p} a + \frac{\varepsilon^{\frac{1}{p-1}}}{p} b^p = a + \frac{(p-1)^{\frac{1}{p-1}}}{p^{\frac{p}{p-1}}} b^p
$$

if we choose $\varepsilon = (p-1)/p$. In this way we can write

$$
I_1 \le \int_\Omega |\nabla u|^p J''(u)\varphi \,\mathrm{d}x + \frac{(p-1)^{\frac{1}{p-1}}}{p^{\frac{p}{p-1}}} \left[\int_\Omega \frac{|J'(u)|^{p\delta'}}{[J''(u)]^{(p-1)\delta'}} \varphi \,\mathrm{d}x\right]^{\frac{1}{\delta'}} \left[\int_\Omega \frac{|\nabla\varphi|^{p\delta}}{\varphi^{p\delta-1}} \,\mathrm{d}x\right]^{\frac{1}{\delta}} \tag{4.37}
$$

All together, we have proved that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega J(u)\varphi \,\mathrm{d}x \le C_1 \left[\int_\Omega \frac{|J'(u)|^{p\delta'}}{[J''(u)]^{(p-1)\delta'}} \varphi \,\mathrm{d}x\right]^{\frac{1}{\delta'}} \left[\int_\Omega \frac{|\nabla\varphi|^{p\delta}}{\varphi^{p\delta-1}} \,\mathrm{d}x\right]^{\frac{1}{\delta}}, \tag{4.38}
$$

with $C_1 = (p-1)^{1/(p-1)}/p^{p/(p-1)}$.

We now specialize J and δ to get the result for $r > 1$. We let $\delta' = r/(r+p-2)$ and $\delta = r/(2-p)$ in (4.38) and estimate the last integral in (4.38) using inequality (4.126) of Lemma 4.6 with $\alpha = pr/(2-p)$, and $C_{2,r}$ that depends only on p, r and n, after choosing the test function φ as there. We obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega J(u)\varphi \,\mathrm{d}x &= C_{2,r} \frac{|B_{R_0} \setminus B_R|^{\frac{2-p}{r}}}{(R_0-R)^p} \left[\int_\Omega \frac{|J'(u)|^{\frac{pr}{r+p-2}}}{[J''(u)]^{\frac{(p-1)r}{r+p-2}}} \varphi \,\mathrm{d}x\right]^{\frac{r+p-2}{r}} \\
&:= C_{3,r}(R_0, R) \left[\int_\Omega \frac{|J'(u)|^{\frac{pr}{r+p-2}}}{[J''(u)]^{\frac{(p-1)r}{r+p-2}}} \varphi \,\mathrm{d}x\right]^{\frac{r+p-2}{r}}.
\end{aligned}
\tag{4.39}
$$

We now choose the convex function $J : [0, +\infty) \to [0, +\infty)$ to be $J(u) = |u|^r$, so that

$$|J'(u)|^{\frac{pr}{r+p-2}}/[J''(u)]^{\frac{(p-1)r}{r+p-2}} = \frac{r^{\frac{pr}{r+p-2}}}{(r-1)^{\frac{(p-1)r}{r+p-2}}} J(u).$$

All together, putting $X(t) = \int_\Omega J(u(\cdot,t))\varphi \,\mathrm{d}x$, we have proved that

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} \le \frac{r^p\, C_{3,r}(R_0,R)}{(r-1)^{p-1}}\, X(t)^{\frac{r+p-2}{r}} := C_{4,r}(R_0,R)\, X(t)^{1-\frac{2-p}{r}}\,. \tag{4.40}$$

We integrate the closed differential inequality (4.40) over (s,t) to obtain

$$\left[\int_\Omega J\big(u(t)\big)\varphi \,\mathrm{d}x\right]^{\frac{2-p}{r}} \le \left[\int_\Omega J\big(u(s)\big)\varphi \,\mathrm{d}x\right]^{\frac{2-p}{r}} + \frac{2-p}{r} C_{4,r}(R_0,R)(t-s).$$

with $C_{4,r}$ as above. We finally estimate both integral terms, using the special form of the test function φ, see e.g. Lemma 4.6, and get

$$\left[\int_{B_R} |u|^r(t)\,\mathrm{d}x\right]^{2-p} \le \left[\int_{B_{R_0}} |u|^r(s)\,\mathrm{d}x\right]^{\frac{2-p}{r}} + C_r(t-s)\,, \tag{4.41}$$

with C_r as in the statement.

It only remains to remove the initial assumption $u(t) \in L^\infty_{loc}$: consider the sequence of essentially bounded functions $u_n(\tau) \to u(\tau)$ in L^r_{loc}, when $n \to \infty$, for a.e. $\tau \in (s,t)$. It is then clear that inequality (4.33) holds for any u_n and we can pass to the limit. □

The reader will notice that the constant C_r above blows up as $r \to 1$, hence the need for a different proof in that limit case.

Theorem 4.11. *Let $u \in C\big((0,T) : W^{1,p}_{loc}(\Omega)\big)$ be a nonnegative bounded local strong solution of the fast p-Laplacian equation, with $1 < p < 2$. Let $0 < R < R_0 \le \operatorname{dist}(x_0, \partial\Omega)$. Then we have*

$$\int_{B_R(x_0)} |u(x,t)|\,\mathrm{d}x \le C_1 \int_{B_{R_0}(x_0)} |u(x,s)|\,\mathrm{d}x + C_2(t-s)^{1/(2-p)}, \tag{4.42}$$

for any $0 \le s \le t \le T$. There C_1 is a constant near 1 that depends on n, p, while C_2 depends also on R and R_0.

Proof. (i) The first part of the proof is identical to the proof of Theorem 4.10 up to formula (4.38), hence we do not repeat it. We proceed then by a different choice of J and δ. We choose λ and ε small in $(0,1)$ and put for all $|u| \ge 1$

$$J'(u) = \operatorname{sign}(u)\left(1 - \frac{\lambda}{(1+|u|)^\varepsilon}\right)$$

while for $|u| \le 1$ we choose a smooth curve that joins smoothly with the previous values. Then we have for $|u| \ge 1$

$$J''(u) = \varepsilon\lambda(1+|u|)^{-1-\varepsilon}, \qquad (1-\lambda)|u| \le J(u) \le |u|.$$

Since $1 < p < 2$ we may always choose ε small enough so that $(p-1)(1+\varepsilon) < 1$. We may then choose $1/\delta' = (p-1)(1+\varepsilon)$ so that $1/\delta = 2-p-\mu$ with $\mu = \varepsilon(p-1)$ also small and positive. In view of the behavior of J, J' and J'' for large $|u|$ we obtain the relation

$$|J'(u)|^{p\delta'}/[J''(u)]^{(p-1)\delta'} \le K_1 J(u) + K_2\,,$$

for some constants K_1 and $K_2 > 0$ that depend only on p, n, ε and λ. Note that K_1 blows up if we try to pass to the limit $\varepsilon \to 0$. Then, (4.38) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J(u)\varphi\,\mathrm{d}x \le C_2\frac{|B_{R_0}\setminus B_R|^{1/\delta}}{(R_0-R)^p}\left[\int_\Omega (K_1J(u)+K_2)\varphi\,\mathrm{d}x\right]^{1/\delta'}, \tag{4.43}$$

where C_2 depends on $p, n,$ and $p\delta$. Therefore, if $Y(t) := \int_\Omega J(u(\cdot,t)\varphi\,\mathrm{d}x$ we get

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} \le C_2\frac{|B_{R_0}\setminus B_R|^{1/\delta}}{(R_0-R)^p}\left(K_1Y(t)+K_2|B_{R_0}|\right)^{1/\delta'} \le C_3\left(Y(t)+C_4\right)^{1/\delta'}, \tag{4.44}$$

where now C_3 and C_4 depend also on R_0, R and δ. Integration of this inequality gives for every $0 < s < t < T$:

$$(Y(t)+C_4)^{1/\delta} \le (Y(s)+C_4)^{1/\delta} + C_5(t-s). \tag{4.45}$$

Since $(1-\lambda)|u| \le J(u) \le |u|$ we easily obtain the basic inequality

$$\left(\int_\Omega J(u(\cdot,t))\varphi\,\mathrm{d}x + C_4\right)^{1/\delta} \le \left(\int_\Omega J(u(\cdot,s))\varphi\,\mathrm{d}x + C_4\right)^{1/\delta} + C_5(t-s). \tag{4.46}$$

(ii) We now translate this inequality into an L^1 estimate. We use the fact that

$$J(u) \le |u| + c_1 \le c_2J(u) + c_3.$$

Therefore, with $a_1 = 1/c_2 = 1-\lambda$ and $a_2 = (c_1-c_3)/c_2$ we have

$$\left(\int_\Omega (a_1|u(\cdot,t)|+a_2)\varphi\,\mathrm{d}x + C_4\right)^{1/\delta} \le \left(\int_\Omega (|u(\cdot,s)|+c_1)\varphi\,\mathrm{d}x + C_4\right)^{1/\delta} + C_5(t-s),$$

that we can rewrite as

$$\left(\int_\Omega (|u(\cdot,t)|+a_2')\varphi\,\mathrm{d}x + C_4'\right)^{1/\delta} \le (1-\lambda)^{1/\delta}\left(\int_\Omega (|u(\cdot,s)|+c_1)\varphi\,\mathrm{d}x + C_4\right)^{1/\delta} + C_5'(t-s).$$

This means that for every $\varepsilon > 0$ we have

$$\left(\int_\Omega |u(\cdot,t)|\,\varphi\,\mathrm{d}x\right)^{1/\delta} \le (1+c(\varepsilon+\lambda))\left(\int_\Omega |u(\cdot,s)|\varphi\,\mathrm{d}x\right)^{1/\delta} + C_\varepsilon + C_5'(t-s).$$

(iii) Let us perform a scaling step. We take a solution u as in the statement and two fixed times $t_1 > t_2 > 0$. We put $h = t_2 - t_1$. We apply now the rule to the rescaled solution $\widehat{u}$ defined as $\widehat{u}(x,t) = h^{-1/(2-p)}u(x, t_1 + th)$ between $s = 0$ and $t = 1$. Then, after raising the expression to the power δ we get

$$\int_\Omega |\widehat{u}(\cdot,1)|\,\varphi\,\mathrm{d}x \le (1 + c'(\varepsilon+\lambda)) \int_\Omega |\widehat{u}(\cdot,0)|\varphi\,\mathrm{d}x + C_6,$$

which implies

$$\int_\Omega |u(\cdot,t_2)|\,\varphi\,\mathrm{d}x \le (1 + c'(\varepsilon+\lambda)) \int_\Omega |u(\cdot,t_1)|\varphi\,\mathrm{d}x + C_6(t_2-t_1)^{1/(2-p)}.$$

We finally eliminate the dependence on ε of the constants by fixing $\varepsilon = (2-p)/2(p-1) > 0$.
□

Remark. In the proofs we use and improve on a technique introduced by Boccardo et al. in [31] to obtain local integral estimates for the p-Laplacian equation in the elliptic framework, both for L^r and L^1 norms, the latter being technically more complicated.

4.3.3 Proof of Theorem 4.8 for bounded strong solutions

We are now ready to prove Theorem 4.8, by joining the space-time smoothing effect and local L^r-norm estimates. We will work with bounded strong solution, but the same proof holds for bounded weak solutions, that are indeed Hölder continuous, thus strong, cf. Appendix A2 and Theorem 4.7. The boundedness assumption will be removed by comparison with suitable extended large solutions in Section 4.5.

Proof. Consider a bounded (hence continuous) local strong solution u defined in $Q_0 = B_{R_0}(x_0) \times (0,T)$, noticing that it is not restrictive to assume $x_0 = 0$. Consider a smaller ball $B_R \subset B_{R_0}$ and take $\rho > 0$, $\varepsilon > 0$ such that $R = \rho(1-\varepsilon)$ and $R_0 = \rho(1+\varepsilon)$. Then we consider the following rescaled solution

$$\tilde{u}(x,t) = Ku(\rho x, \tau t), \qquad K = \left(\frac{\rho^p}{\tau}\right)^{\frac{1}{2-p}}, \qquad \tau \in (0,T), \tag{4.47}$$

and we apply the result of Theorem 4.9 to the solution $\tilde{u}$ in the cylinders $\widetilde{Q}_0 = B_1 \times [0,1]$ and $\widetilde{Q} = B_{1-\varepsilon} \times [\varepsilon^p, 1]$, for some $\varepsilon \in (0,1)$. Recalling the notation $q = n/p$, we obtain

$$\sup_{\widetilde{Q}} |\tilde{u}| \le \frac{C(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}} \left[\iint_{\widetilde{Q}_0} \tilde{u}^r\,\mathrm{d}x\,\mathrm{d}t + \omega_n\right]^{\frac{1}{r+(p-2)q}}. \tag{4.48}$$

We then use Theorem 4.10 for $r \ge 1$ on the balls $B_1 \subset B_{1+\varepsilon}$

$$\int_{B_1} |\tilde{u}(x,t)|^r\,\mathrm{d}x \le 2^{\frac{r}{2-p}-1}\left[\int_{B_{1+\varepsilon}} |\tilde{u}(x,0)|^r\,\mathrm{d}x + \big(C_r(1,1+\varepsilon,p,n)\,t\big)^{\frac{r}{2-p}}\right], \tag{4.49}$$

where we use the inequality $(a+b)^l \le 2^{l-1}(a^l+b^l)$ for $l = r/(2-p) > 1$. The constant is

$$C_r(1,1+\varepsilon,p,n) = \frac{C(r,p,n)}{\varepsilon^p}|B_{1+\varepsilon}\setminus B_1|^{\frac{2-p}{r}} \le C(r,p,n)\varepsilon^{\frac{2-p}{r}-p}, \text{ if } r>1\,,$$
$$C_r(1,1+\varepsilon,p,n) = \frac{C(p,n)}{\varepsilon^p}|B_{1+\varepsilon}\setminus B_1|^{2-p} + |B_{1+\varepsilon}|^{2-p}, \text{ if } r=1.$$

We integrate in time over $(0,1)$ and we obtain:

$$\iint_{\widetilde{Q}_0} \tilde{u}^r \,\mathrm{d}x\,\mathrm{d}t \le 2^{\frac{r}{2-p}-1}\left[\int_{B_{1+\varepsilon}} |\tilde{u}(x,0)|^r \,\mathrm{d}x + \frac{1}{\frac{r}{2-p}+1} C_r(1,1+\varepsilon,p,n)^{\frac{r}{2-p}}\right].$$

We now join the previous estimates and we obtain:

$$\begin{aligned}
\sup_{x\in B_{1-\varepsilon},\, t\in[\varepsilon^p,1]} \tilde{u}(x,t) &\le \frac{C(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}}\left\{\left[2^{\frac{r}{2-p}-1}\int_{B_{1+\varepsilon}} \tilde{u}(x,0)^r \,\mathrm{d}x\right]^{\frac{1}{r+(p-2)q}}\right.\\
&\quad \left. + \left[\omega_n + 2^{\frac{r}{2-p}-1}\frac{C_r(1,1+\varepsilon,p,n)}{1+\frac{r}{2-p}}\right]^{\frac{1}{r+(p-2)q}}\right\}\\
&= \widetilde{C}_{1,\varepsilon}\left[\int_{B_{1+\varepsilon}} \tilde{u}(x,0)^r \,\mathrm{d}x\right]^{\frac{1}{r+(p-2)q}} + \widetilde{C}_{2,\varepsilon}.
\end{aligned}$$

Then we rescale back from $\tilde{u}$ to the initial solution u. From the last estimate, we get

$$\sup_{x\in B_{1-\varepsilon},\, t\in[\varepsilon^p,1]} Ku(x\rho,\tau t) \le \widetilde{C}_{1,\varepsilon}K^{\frac{r}{r+(p-2)q}}\left[\int_{B_{1+\varepsilon}} \tilde{u}(x,0)^r \,\mathrm{d}x\right]^{\frac{1}{r+(p-2)q}} + \widetilde{C}_{2,\varepsilon},$$

or, after setting $s = \tau t$ and $y = x\rho$, we can write equivalently:

$$\sup_{y\in B_{(1-\varepsilon)\rho},\, s\in(\tau\varepsilon^p,\tau)} u(y,s) \le \widetilde{C}_{1,\varepsilon}K^{\frac{r}{r+(p-2)q}-1}\rho^{-\frac{n}{r+(p-2)q}}\left[\int_{B_{(1+\varepsilon)\rho}} u_0(y)^r \,\mathrm{d}y\right]^{\frac{1}{r+(p-2)q}} + \frac{\widetilde{C}_{2,\varepsilon}}{K}.$$

Replacing K with ρ and τ as in (4.47), we see that the term in ρ disappears, so that

$$\sup_{y\in B_{(1-\varepsilon)\rho},\, s\in(\tau\varepsilon^p,\tau)} u(y,s) \le \frac{\widetilde{C}_{1,\varepsilon}}{\tau^{\frac{n}{rp+(p-2)n}}}\left[\int_{B_{(1+\varepsilon)\rho}} |u_0(x)|^r \,\mathrm{d}x\right]^{\frac{p}{rp+(p-2)n}} + \widetilde{C}_{2,\varepsilon}\left(\frac{\tau}{\rho^p}\right)^{\frac{1}{2-p}}.$$

We finally let $t=\tau$, $R_0 = \rho(1+\varepsilon)$, $R = \rho(1-\varepsilon)$, $C_1 = \widetilde{C}_{1,\varepsilon}$, $C_2 = \widetilde{C}_{2,\varepsilon}$ and replace q by its value n/p, in order to get the notations of Theorem 4.8. The proof of the main quantitative estimate (4.16) is concluded once we analyze the constants

$$C_1 = \frac{K_1(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}}, \quad C_2 = \frac{K_2(r,p,n)}{\varepsilon^{\frac{p(q+1)}{r+(p-2)q}}}\left[K_3(r,p,n)\varepsilon^{\left(\frac{2-p}{r}-p\right)\frac{r}{2-p}} + K_4(p,n)\right]^{\frac{1}{r+(p-2)q}},$$

where $K_i(r,p,n)$, $i = 1,2,3$, are positive constants independent on ε, and

$$K_4(p,n) = \omega_n \text{ if } r > 1, \quad K_4(p,n) = \omega_n + \frac{2-p}{3-p}\, 2^{n(2-p)+\frac{p-1}{2-p}}, \text{ if } r = 1.$$

We conclude by letting $\varepsilon = (R_0 - R)/(R_0 + R)$ in the formulas above. $\square$

4.4 Large solutions for the parabolic p-Laplacian equation

We call *continuous large solution* of the p-Laplacian equation, a function u solving the following boundary problem

$$\begin{cases} u_t = \Delta_p u\,, & \text{in } Q_T, \\ u(x,t) = +\infty\,, & \text{on } \partial\Omega \times (0,T), \\ u(x,t) < +\infty\,, & \text{in } Q_T, \end{cases}$$

in the sense that u is satisfies the local weak formulation (4.2) in the cylinder $Q_T = \Omega \times (0,T)$, where Ω is a domain in $\mathbb{R}^n$, is continuous in Q_T, and it takes the boundary data in the continuous sense, that is $u(x,t) \to +\infty$ as $x \to \partial\Omega$. Note that there is no reference to the initial data in this definition. If initial data are given, they will be taken as initial traces as mentioned before. In the sequel we will assume that Ω is bounded and has a smooth boundary but such requirement is not essential and is done here for the sake of simplicity.

Using the results of Theorem 4.1 we are ready to establish the existence of large solutions for general bounded domains Ω. We have the following:

Theorem 4.12. *Let either $1 < p \le p_c$ and $r > r_c$, or $p_c < p < 2$ and $r \ge 1$. Given $u_0 \in L^r_{loc}(\Omega)$, there exists a continuous large solution of the p-Laplacian equation in Ω having u_0 as initial data. Such solutions are moreover Hölder continuous in the interior, and satisfy the local smoothing effect of Theorem 4.1.*

Proof. We obtain first the solution by an approximation procedure. We consider the following Dirichlet problem:

$$(P_n) \begin{cases} u_t = \Delta_p u, \text{ in } Q_T, \\ u(x,t) = n, \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \text{ in } \Omega, \end{cases} \tag{4.50}$$

which admits a unique continuous weak solution u_n, by well established theory (see e.g. [52]). It is easy to observe that the unique solution u_n of (P_n) becomes a subsolution for the problem (P_{n+1}). Since any subsolution is below any solution of the standard Dirichlet problem, we find that $u_n \le u_{n+1}$ in Q_T. By monotonicity we can therefore define the pointwise limit $u(x,t) = \lim\limits_{n\to\infty} u_n(x,t)$. Moreover, u_n satisfies the local bounds for the gradient, Theorem 4.17, since any weak solution is in particular a local weak solution. Using the energy estimates of Theorem 4.17, it is then easy to check that the sequence $\{|\nabla u_n|\}$ is uniformly bounded in $L^p_{loc}(Q_T)$, independently on n, hence it converges weakly in this space to a function v.

By standard arguments $v = \nabla u$. Next, we write the local weak formulation for u_n, on any compact $K \times [t_1, t_2] \subset Q_T$:

$$\int_K u_n(t_2)\varphi(t_2)\ \mathrm{d}x - \int_K u_n(t_1)\varphi(t_1)\ \mathrm{d}x = -\int_{t_1}^{t_2} (u_n\varphi_t + |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi)\ \mathrm{d}x\ \mathrm{d}t,$$

for any test function as in Definition 4.1. We can pass to the limit as $n \to \infty$ by the previous observations and the monotone convergence theorem, so that the limit u satisfies the local weak formulation (4.2). From our local smoothing effect and Dini's Theorem, we deduce that $u_n \to u$ locally uniformly.

Moreover, $u(x,t) \to +\infty$ as $x \to \partial\Omega$; the fact that the boundary data is taken in the continuous sense follows from comparison with the solution of the same problem with initial data 0, which has the separate variables form and takes boundary data in the continuous sense, cf. [50]. The last condition is that $u(x,t) < +\infty$ in Q_T; but this follows directly from Theorem 4.1 by our assumptions. Hence, u is a Hölder continuous large solution for the p-Laplacian equation. □

Remark. Large solutions are a typical feature of fast diffusion equations. We recall that in the case of the fast diffusion equation $u_t = \Delta u^m$ with $0 < m < 1$, the theory of large solutions can be developed as a particular case of the theory of solutions with general Borel measures as initial data constructed by Chasseigne and Vázquez in [40] with the name of *extended continuous solutions*. The existence and uniqueness of large solutions has been completely settled in that paper for $m_c = (n-2)/n < m < 1$. For $m < m_c$, a general uniqueness result of such solutions is still open. A similar approach can be applied to the fast p-Laplacian equation considered in the present chapter, but the detailed presentation entails modifications that deserve a careful presentation.

We next establish a sharp space-time asymptotic estimate, which also gives the blow-up rate of large solutions near the boundary or for large times.

Theorem 4.13. *Let u be a continuous large solution with initial datum u_0, in the conditions of Theorem 4.12. We have the following bounds:*

$$\frac{C_0\, t^{\frac{1}{2-p}}}{\mathrm{dist}(x,\partial\Omega)^{\frac{p}{2-p}}} \le u(x,t) \le \frac{C_1\, t^{\frac{1}{2-p}}}{\mathrm{dist}(x,\partial\Omega)^{\frac{p}{2-p}}} + C_2, \tag{4.51}$$

for some positive constants C_0, C_1 and C_2. In particular $u = O(\mathrm{dist}(x,\partial\Omega)^{\frac{p}{p-2}})$ as $x \to \partial\Omega$.

Proof. The upper bound comes from a direct application of the Local Smoothing Effect, Theorem 4.1. For the lower bound, we compare with the continuous large solution with initial datum $u_0 \equiv 0$. We look for a separate variable solution of the form $u(x,t) = \phi(x)t^{1/(2-p)}$, hence ϕ is a large solution of the elliptic problem:

$$\begin{cases} \Delta_p\phi = \lambda\phi, & \text{in } \Omega \\ \phi = +\infty & \text{on } \partial\Omega. \end{cases}$$

Analyzing this problem for a ball $\Omega = B_R$, we find that there exists a unique radial large solution, namely

$$u(x,t) = k(p)\,\frac{t^{\frac{1}{2-p}}}{d(x)^{\frac{p}{2-p}}}, \qquad k(p)^{2-p} = \frac{2(p-1)p^{p-1}}{(2-p)^p}\,,$$

where $d(x) = R - |x|$. This precise expression does not depend on the radius of the ball, and it is in fact true to first approximation for the large solution of the elliptic problem in any bounded domain with a C^1 boundary, cf. [50]. $\square$

The existence and properties of large solutions will be used to conclude the proof of Theorem 4.8. Such conclusion consists in passing from a bounded local strong solution to a general local strong solution. This will be done essentially by showing that any local strong solution can be bounded above by a large solution in a small ball around the point under consideration, with the same local initial trace u_0. The difficult technical problem is that we have to take into account the boundary data in the comparison. The way out of this difficulty is a modification of the construction of large solutions that leads to the concept of "extended large solutions". Such ideas are originated in [40] for the fast diffusion equation.

Extended large solutions. We now present an alternative approach to the construction of continuous large solutions that will be needed in the sequel to establish some technical results. We will only need the construction on a ball. Take $0 < R < R_1$, let $B_R \subset B_{R_1} \subset \Omega$ and $A = B_{R_1} \setminus B_R$, and consider the following family of Dirichlet problems

$$(\mathbb{D}_n) \qquad \begin{cases} \partial_t v_n = \Delta_p v_n, & \text{in } B_{R_1} \times (0,T), \\ v_n(x,t) = n, & \text{on } \partial B_{R_1} \times (0,T), \\ v_n(x,0) = \begin{cases} u_0(x), \\ n, \end{cases} & \begin{matrix} \text{in } B_R, \\ \text{in } A. \end{matrix} \end{cases}$$

Let $v_n(x,t)$ be the unique, continuous local strong solution to the above Dirichlet problem, corresponding to the initial datum $u_0 \in L^r_{loc}(B_R)$. Such solutions exist for all $0 < t < \infty$ and form a family of locally bounded solutions that satisfy the local smoothing effect of Theorem 4.8, since they are continuous. We stress that the initial datum v_0 need not to have the gradient well defined on B_R, but in the annulus A we have $\nabla v_0 \equiv 0$. As in the proof of Theorem 4.12, we see that the sequence $\{v_n\}$ is monotone increasing, $v_n(x,t) \le v_{n+1}(x,t)$ for a.e. (x,t), and converges pointwise to a function V which is a solution of the fast p-Laplacian equation in $B_R \subset B_{R_1}$, and that we will call *extended large solution.* We next investigate the behavior of the extended large solution V in the annulus $A = B_{R_1} \setminus B_R$.

Proposition 4.1. *Under the running assumptions on v_n and V, The extended large solution satisfies*

(i) The restriction of V to B_R is a continuous large solution in the sense specified at the beginning of this section, and of Theorem 4.12.

(ii) V is "large" when extended to the annulus A, in the sense that

$$V(x,t) = \lim_{n\to\infty} v_n(x,t) = +\infty \qquad \textit{for all } \ (x,t) \in A \times (0,T)$$

and the divergence is uniform.

(iii) The initial trace $V_0 := \lim_{t\to 0^+} V(t,\cdot) = u_0$ in B_R, while $V_0 = +\infty$ in A.

Remark. The above result somehow proves the sharpness of Theorem 4.12 and motivates the terminology "extended large solution". Obviously, V_0 is not in L^r_{loc}, and the smoothing effect cannot hold in A.

Proof. We only need to prove (i) and (ii), since (iii) easily follows by construction. Parts (i) and (ii) follow from local L^1 estimates together with a comparison with suitable radially symmetric subsolutions.

Radially symmetric subsolutions. We define a special class of subsolutions $\widetilde{v_n}$: consider the problem $(\mathbb{D}_n)$, repeat the same construction made for v_n, but now we choose $u_0 = 0$ in B_R. Obviously, $\widetilde{v_n} \le v_n$ in B_{R_1}, $\widetilde{v_n} \le \widetilde{v_{n+1}}$, and they are all radially symmetric. Moreover, by the maximum principle we know that each function $\widetilde{v_n}$ is nondecreasing along the radii, thus

$$\int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x \le \widetilde{v_n}(\overline{x},t) \int_{B_r(x_0)} \varphi(x)\,\mathrm{d}x, \tag{4.52}$$

where $\overline{x}$ is the point of $\overline{B_r(x_0)}$ with maximum modulus, since $\widetilde{v_n}$ is radially symmetric in the bigger ball B_{R_1} and $\varphi \ge 0$ is a suitable test function that will be chosen later.

L^1 estimates. These estimates are possible thanks to the local L^p bounds (4.119) valid for the gradient of the solution $\widetilde{v_n}$ to the Dirichlet problems $\mathbb{D}_n$, namely, for any small ball $B_{r+\varepsilon}(x_0) \subset A$ and

$$\int_{B_r(x_0)} |\nabla \widetilde{v_n}(x,t)|^p \,\mathrm{d}x \le c_0 \int_{B_{r+\varepsilon}(x_0)} |\nabla \widetilde{v_n}(x,0)|^p \,\mathrm{d}x + c_1 t^{\frac{p}{2-p}} = c_1 t^{\frac{p}{2-p}}\,, \tag{4.53}$$

the last equality holds since by definition the gradient of the initial data is zero in A.

We now fix a time $t \in (0,T]$, a point $x_0 \in A$ and a ball $B_{r+\varepsilon}(x_0) \subset A$. We choose a suitable test function φ supported in $B_r(x_0)$, and we calculate

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x\right| &= \left|\int_{B_r(x_0)} \Delta_p\big(\widetilde{v_n}(x,t)\big)\varphi(x)\,\mathrm{d}x\right| \\
&= \left|-\int_{B_r(x_0)} \big|\nabla\widetilde{v_n}(x,t)\big|^{p-2}\nabla\widetilde{v_n}(x,t)\cdot\nabla\varphi(x)\,\mathrm{d}x\right| \\
&\le \int_{B_r(x_0)} \big|\nabla\widetilde{v_n}(x,t)\big|^{p-1}\big|\nabla\varphi(x)\big|\,\mathrm{d}x \\
&\le \left[\int_{B_r(x_0)} \big|\nabla\varphi(x)\big|^p\,\mathrm{d}x\right]^{\frac{1}{p}} \left[\int_{B_r(x_0)} \big|\nabla\widetilde{v_n}(x,t)\big|^p\,\mathrm{d}x\right]^{\frac{p-1}{p}} \\
&\le C_\varphi\, c_1 t^{\frac{p-1}{2-p}} := C\, t^{\frac{p-1}{2-p}},
\end{aligned} \tag{4.54}$$

where in the second line we performed an integration by parts that can be justified in view of the Hölder regularity of the solution and by Corollary 4.1. In the fourth line we have used Hölder inequality, and in the last step the inequality (4.53) and the fact that the integral of the test function is bounded. We integrate such differential inequality over $(0,t)$ to get

$$\left|\int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x - \int_{B_r(x_0)} \widetilde{v_n}(x,0)\varphi(x)\,\mathrm{d}x\right| \le C\,t^{\frac{1}{2-p}}. \tag{4.55}$$

Taking into account that $\widetilde{v_n}(x,0) = n$ and (4.52), we obtain

$$\widetilde{v_n}(\overline{x},t)\int_{B_r(x_0)} \varphi(x)\,\mathrm{d}x \ge \int_{B_r(x_0)} \widetilde{v_n}(x,t)\varphi(x)\,\mathrm{d}x \ge n\int_{B_r(x_0)} \varphi(x)\,\mathrm{d}x - C\,t^{\frac{1}{2-p}}\,, \tag{4.56}$$

hence $\widetilde{v_n}(\overline{x},t) \to \infty$ as $n \to \infty$, since $\overline{x}$ does not depend on n. Since $\widetilde{v_n}$ is radially symmetric, we have proved that $\widetilde{v_n}(x,t) \to \infty$ as $n \to \infty$ for any $|x| = |\overline{x}|$. We can repeat the argument for any small ball $B_r(x_0) \subset A$, and we obtain that $\widetilde{v_n}(x,t) \to \infty$ for any $x \in A$ and $t > 0$, but not for $|x| = R$. This result extends to $v_n \ge \widetilde{v_n}$ by comparison.

Behaviour of V in $\overline{B_R}$. Let $0 < R < R' < R_1$ and let $L_{R'}$ be the continuous large solution in $B_{R'}$ whose initial trace is 0 in $B_{R'}$. Since $L_{R'}$ satisfies the local smoothing effect (4.16), we can compare it on a smaller ball say $B_{R'-\varepsilon}$, with a suitably chosen $\widetilde{v_n}$, namely

$$\forall\varepsilon > 0 \qquad \exists n_\varepsilon \quad \text{such that} \quad \widetilde{v_{n_\varepsilon}}(x,t) \ge L_{R'}(x,t) \quad \text{for any } (x,t) \in B_{R'-\varepsilon} \times (0,T]\,.$$

This implies that $\widetilde{V} := \lim_{n\to\infty} \widetilde{v_n} \ge L_{R'}$ in $B_{R'-\varepsilon} \times (0,T]$ for any $\varepsilon > 0$. Letting now $\varepsilon \to 0$, we obtain that $\widetilde{V} \ge L_{R'}$ in $B_{R'} \times (0,T]$ and this holds for any $R' \in (R, R_1)$.

By scaling we can identify different continuous large solutions in different balls, namely let L_R and $L_{R'}$ be the large solutions corresponding to the balls $B_R \subset B_{R'}$, and

$$L_{R'}(x,t) = L_{R,\lambda}(x,t) := \lambda^{\frac{p}{2-p}} L_R\big(\lambda x, t\big), \qquad \text{with } \lambda = \frac{R}{R'} < 1.$$

It is then clear that $L_{R'} \to L_R$ when $R' \to R$ at least pointwise in $\overline{B_R} \times (0,T]$, and this implies also that $\widetilde{V} \ge L_R$ in $\overline{B_R} \times (0,T]$ and in particular

$$\lim_{x\to\partial B_R} \widetilde{V}(x,t) \ge \lim_{x\to\partial B_R} L_R(x,t) = +\infty \qquad \text{in the continuous sense.}$$

By comparison, we see that $V \ge \widetilde{V}$, hence $\lim_{x\to\partial B_R} \widetilde{V}(x,t) = +\infty$ in the continuous sense. The initial trace of V in B_R is $u_0 \in L^r_{loc}$, thus the local smoothing effect applies and implies, as usual, that V is locally bounded in B_R, therefore it is continuous. The proof is concluded since we have proved that V is an extended large solution, in the above sense. □

The uniqueness of the extended large solution is a delicate matter in general. It is easy to show uniqueness of such solutions in a ball, but a complete result is not known. We will not tackle this problem here.

4.5 Local boundedness for general strong solutions. End of proof of Theorem 4.8

Let us now conclude the proof of Theorem 4.8. The last step in the proof consists in comparing a general (non necessarily bounded) local strong solution u with the extended large solution V that is known to satisfy the smoothing effect (4.16).

Let u be the local strong solution, $u_0 \in L^r_{loc}$ be its initial trace, as in the assumption of Theorem 4.8. The comparison $u \le V$ will be proved through an approximated L^1 contraction principle, which uses the approximating sequence v_n defined above. We borrow some ideas from Proposition 9.1 of [131]. Let us introduce a function $P \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, such that $P(s) = 0$ for $s \le 0$, $P^{'}(s) > 0$ for $s > 0$ which is a smooth approximation of the positive sign function

$$\mathrm{sgn}^+(s) = 1 \text{ if } s > 0, \qquad \mathrm{sgn}^+(s) = 0 \text{ if } s \le 0.$$

The primitive $Q(s) = \int\limits_0^s P(t)\ \mathrm{d}t$, is an approximation of the positive part: $Q(s) \sim [s]^+$.

Proposition 4.2. *Under the running notations and assumptions, the following "approximate L^1 contraction principle" holds:*

$$\int\limits_{B_R} [u(x,t) - v_n(x,t)]_+ \ \mathrm{d}x \le \int\limits_{B_{R_1}} [u(x,s) - v_n(x,s)]_+ \ \mathrm{d}x + C_n, \tag{4.57}$$

where $C_n \to 0$ as $n \to \infty$.

Proof. We choose a function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ in $B_{R+\varepsilon} \subset B_{R_1}$, $\operatorname{supp}\varphi \subset B_{R_1}$ and $0 \le \varphi \le 1$. We calculate:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{B_{R_1}} Q(u - v_n)\varphi \ \mathrm{d}x &= \int\limits_{B_{R_1}} Q^{'}(u - v_n)(\Delta_p u - \Delta_p v_n)\varphi \ \mathrm{d}x \\
&= \int\limits_{B_{R_1}} P(u - v_n)\, \mathrm{div}(|\nabla u|^{p-2}\nabla u - |\nabla v_n|^{p-2}\nabla v_n)\varphi \ \mathrm{d}x \\
&= -\int\limits_{B_{R_1}} P^{'}(u - v_n)(\nabla u - \nabla v_n)\cdot(|\nabla u|^{p-2}\nabla u - |\nabla v_n|^{p-2}\nabla v_n)\varphi \ \mathrm{d}x \\
&\quad - \int\limits_{B_{R_1}} P(u - v_n)(|\nabla u|^{p-2}\nabla u - |\nabla v_n|^{p-2}\nabla v_n)\cdot\nabla\varphi \ \mathrm{d}x = I_1 + I_2,
\end{aligned}$$

where the calculations are allowed since u and v_n are both local strong solutions. Taking into account the monotonicity of the p-Laplace operator and the fact that $P^{'} \ge 0$, we obtain that $I_1 \le 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{B_{R_1}} Q(u - v_n)\varphi \ \mathrm{d}x \le \int\limits_{A_\varepsilon} P(u - v_n)(|\nabla u|^{p-1} + |\nabla v_n|^{p-1})\, |\nabla\varphi| \ \mathrm{d}x = I_3 + I_4, \tag{4.58}$$

since $\operatorname{supp}\nabla\varphi \subset A_\varepsilon := B_{R_1} \setminus B_{R+\varepsilon}$. We then have:

$$I_3 := \int_{A_\varepsilon} P(u-v_n)|\nabla u|^{p-1}\,|\nabla\varphi|\;\mathrm{d}x \to 0, \quad \text{as } n\to\infty,$$

since $u(t) \in W^{1,p}_{loc}(\Omega)$ for any $t>0$, and $P(u-v_n)\to 0$ by construction. Moreover

$$I_4 := \int_{A_\varepsilon} P(u-v_n)|\nabla v_n|^{p-1}\,|\nabla\varphi|\;\mathrm{d}x \le \mathcal{K}(R)\left(\int_{A_\varepsilon} P(u-v_n)^p\;\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{A_\varepsilon}|\nabla v_n|^p\;\mathrm{d}x\right)^{\frac{p-1}{p}}.$$

Using the gradient inequality (4.53) and the fact that $P(u-v_n)\to 0$ a. e., we obtain that $I_4 \to 0$ as $n\to\infty$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{R_1}} Q(u-v_n)\varphi\;\mathrm{d}x \le \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n\to\infty$. An integration of the above differential inequality on (s,t), gives

$$\int_{B_{R_1}} Q\big(u(x,t)-v_n(x,t)\big)\varphi(x)\;\mathrm{d}x - \int_{B_{R_1}} Q\big(u(x,t)-v_n(x,t)\big)\varphi(x)\;\mathrm{d}x \le \varepsilon_n(t-s)\,.$$

Letting P tend to sgn^+ and Q to $[s]^+$, and taking into account the special choice of φ, we obtain

$$\int_{B_R}[u(x,t)-v_n(x,t)]_+\;\mathrm{d}x \le \int_{B_{R_1}}[u(x,s)-v_n(x,s)]_+\;\mathrm{d}x + \varepsilon_n(t-s), \tag{4.59}$$

for any $0\le s\le t<T$. Since $\varepsilon_n(t-s)\le T\varepsilon_n$, we have proved (4.57) with $C_n = T\varepsilon_n$. □

We put $s=0$ in (4.57), recalling that $v_n(x,0)=u_0$, and we pass to the limit as $n\to\infty$ in the left-hand side of (4.57), to find

$$\int_{B_R}[u(x,t)-V(x,t)]_+\;\mathrm{d}x \le 0, \tag{4.60}$$

hence $u(x,t)\le V(x,t)$ for a. e. $(x,t)\in B_R$.

Since V is locally bounded in B_R, it satisfies the local smoothing effect (4.16) in B_R, and $V_0 = u_0$ in B_R. The smoothing effect (4.16) then holds for any local strong solution u with initial trace $u_0 \in L^r_{loc}$. This concludes the proof of Theorem 4.8. □

Remarks. (i) A posteriori, we can "close the circle" by proving that indeed *any local strong solution u with initial trace $u_0\in L^r_{loc}$, is Hölder continuous* (cf. Appendix A2), since it is locally bounded via the local smoothing effect of Theorem 4.8.

(ii) The same proof applies to nonnegative strong subsolutions as in Definition 4.1, hence the upper bound (4.16) holds for initial traces with any sign, not only for nonnegative. This can be done by repeating the whole proof, replacing the local strong solution u and its initial trace u_0 with the nonnegative strong subsolution u^+ and its trace u_0^+ respectively.

4.6 Positivity for a minimal Dirichlet problem

We follow the strategy introduced in [34] for the fast-diffusion equation to prove quantitative lower bounds for a suitable Dirichlet problem. More specifically, we will consider what we call "minimal Dirichlet problem", MDP in the sequel, whose nonnegative solutions lie below any nonnegative continuous local weak solution. As a by-product of the concept of local weak solution, the estimates can be extended to continuous weak solutions to any other problem, such as Neumann, Dirichlet (even non homogeneous or large), Robin, Cauchy, or any other initial-boundary problem on any (even unbounded) domain Ω containing $B_{R_0}(x_0)$. Let us introduce the *Minimal Dirichlet Problem*

$$\text{(MDP)} \quad \begin{cases} u_t = \Delta_p u, \ \text{in } B_{R_0} \times (0,T), \\ u(x,0) = u_0(x), \ \text{in } B_{R_0}, \quad \operatorname{supp}(u_0) \subseteq B_R(x_0) \\ u(x,t) = 0, \ \text{for } t > 0 \text{ and } x \in \partial B_{R_0}, \end{cases} \tag{4.61}$$

where $B_{R_0} = B_{R_0}(x_0) \subset \mathbb{R}^n$, and $0 < 2R < R_0$. The properties of existence and uniqueness for this problem are well-known, in particular, for any initial data $u_0 \in L^2(B_{R_0})$, the problem admits a unique weak solution $u \in C\big([0,\infty) : L^2(B_{R_0})\big) \cap L^p\big((0,\infty) : W_0^{1,p}(B_{R_0})\big)$, cf. [52].

In the range $1 < p < 2$ any such solution of (4.61) extinguishes in finite time; we denote the finite extinction time by $T = T(u_0)$. In general it is not possible to have an explicit expression for $T(u_0)$ in terms of the data, but we have lower and upper estimates for T, cf. (4.124) and Subsection 4.7.3 below.

Let u_D be the solution to the MDP posed on a ball $B_{R_0} \subset \Omega$, and let T_D be its finite extinction time. A priori we can not compare u_D with any local weak solution $u \geq 0$, because the parabolic boundary data can be discontinuous. We therefore restrict u to the class of bounded (hence continuous) local weak solutions and we can compare u with u_D, to conclude that *any solution of the MDP lies below any nonnegative and continuous local weak solution, with the same initial trace on the smaller ball B_R*. As a by-product of this comparison, if the local weak solution also has an extinction time T, then we have $T_D \leq T$, for this reason we have called T_D *minimal life time* for the general local weak solution.

4.6.1 The Flux Lemma

In the previous MDP all the initial mass is concentrated in a smaller ball B_R. The next result explains in a quantitative way how in this situation the mass is transferred to the annulus $B_{R_0} \setminus B_R$ across the internal boundary ∂B_R. Throughout this subsection we will set $A_1 := B_{R_0} \setminus B_R$ and we will consider a cutoff function φ supported in B_{R_0} and taking the value 1 in $B_R \subset B_{R_0}$.

Lemma 4.4. *Let u be a continuous local weak solution to the MDP (4.61) and let φ be a suitable cutoff function as above. Then the following equality holds:*

$$\int_{B_{R_0}} u(x,s)\varphi(x)\,\mathrm{d}x = \int_s^T \int_{A_1} |\nabla u(x,\tau)|^{p-2} \nabla u(x,\tau) \cdot \nabla\varphi(x)\,\mathrm{d}x\,d\tau, \tag{4.62}$$

for any $s \in [0,T]$. In particular, eliminating the dependence on φ, we obtain the following estimate:

$$\int_{B_R} u(x,s)\,\mathrm{d}x \le \frac{k}{R_0-R}\int_s^T\int_{A_1} |\nabla u(x,\tau)|^{p-1}\,\mathrm{d}x\,d\tau, \tag{4.63}$$

for a suitable constant $k = k(n)$ and for any $s \in [0,T]$.

Proof. Let $0 \le s \le t \le T$. We begin by calculating

$$\begin{aligned}\int_s^t\int_{A_1} u_t\varphi\,\mathrm{d}x\,\mathrm{d}\tau &= \int_s^t\int_{A_1} \mathrm{div}(|\nabla u|^{p-2}\nabla u)\varphi\,\mathrm{d}x\,\mathrm{d}\tau\\ &= -\int_s^t\int_{A_1} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau + \int_s^t\int_{\partial B_R} |\nabla u|^{p-2}\,(\partial_\nu u)\,\varphi\,d\sigma\,\mathrm{d}\tau\\ &\quad+\int_s^t\int_{\partial B_{R_0}} |\nabla u|^{p-2}\,(\partial_\nu u)\,\varphi\,d\sigma\,\mathrm{d}\tau,\end{aligned}$$

where ν is the outward normal vector to the boundary of the annulus A_1. Since $\varphi = 0$ on ∂B_{R_0}, the last integral above vanishes. By integrating the left-hand side and taking into account that $\varphi = 1$ on ∂B_R, we obtain:

$$\int_{A_1} u(t)\varphi\,\mathrm{d}x - \int_{A_1} u(s)\varphi\,\mathrm{d}x = -\int_s^t\int_{A_1} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau + \int_s^t\int_{\partial B_R} |\nabla u|^{p-2}\partial_\nu u\,d\sigma\,\mathrm{d}\tau.$$

We put in this equality $t = T$, the finite extinction time of the solution of (4.61), hence we have:

$$\int_{A_1} u(s)\varphi\,\mathrm{d}x = \int_s^T\int_{A_1} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau - \int_s^T\int_{\partial B_R} |\nabla u|^{p-2}\partial_\nu u\,d\sigma\,\mathrm{d}\tau. \tag{4.64}$$

On the other hand, we calculate the same quantity inside the small ball B_R. Since $\varphi \equiv 1$ in B_R, we can omit the test function here. We obtain:

$$\int_{B_R} \big[u(t)-u(s)\big]\,\mathrm{d}x = \int_s^t\int_{B_R} \mathrm{div}(|\nabla u|^{p-2}\nabla u)\,\mathrm{d}x\,\mathrm{d}\tau = \int_s^t\int_{B_R} |\nabla u|^{p-2}\partial_{\nu^*}u\,\mathrm{d}x\,\mathrm{d}\tau,$$

where we denote by ν^* the outward normal vector to the boundary of the ball B_R. Then $\nu^* = -\nu$, hence $\partial_\nu u = -\partial_{\nu^*}u$. Letting again $t = T$, we get

$$\int_{B_R} u(x,s)\,dx = \int_s^T\int_{\partial B_R} |\nabla u|^{p-2}\partial_\nu u\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{4.65}$$

Joining relations (4.64) and (4.65), we see that the terms on the boundary compensate, the flux going out of the ball B_R across its boundary equals the flux entering A_1. By canceling these flux terms, we obtain exactly the identity (4.62). In order to get the estimate (4.63), it suffices now to remark that, since $\mathrm{supp}\varphi \subset B_{R_0}$ and $\varphi \equiv 1$ in B_R, then there exists a choice of φ and an universal constant $k = k(n)$, depending only on the dimension, such that

$$|\nabla\varphi(x)| \le \frac{k(n)}{R_0-R},$$

for any $x \in A_1$. This concludes the proof. □

Remark: Note that the undesired boundary term is eliminated only by the fact that $\varphi = 0$ on ∂B_{R_0}, independently of u. Hence, the same estimates (4.62) and (4.63) are true in any balls $B_R \subset B_{r_1} \subset B_{r_2} \subset B_{R_0}$, the only difference in the proof being the choice of φ.

A local Aleksandrov reflection principle. Here we state the Aleksandrov reflection principle in the version adapted for the minimal Dirichlet problem (4.61). That is:

Proposition 4.3. *Let u be a continuous local weak solution to the MDP* (4.61). *Then, for any $t > 0$, we have $u(x_0, t) \geq u(x, t)$, for any $t > 0$ and $x \in A_2 := B_{R_0}(x_0) \setminus B_{2R}(x_0)$. In particular, this implies the following mean-value inequality:*

$$u(x_0, t) \geq \frac{1}{|A_2|} \int_{A_2} u(x, t)\, dx. \tag{4.66}$$

In other words, this inequality says that the mean value of the solution of (4.61) in an annulus is less than the value at the center of the ball where the whole mass was concentrated at the initial time. The proof is a straightforward adaptation of the proof of the corresponding local Aleksandrov principle for the fast diffusion equation, given by two of the authors in [35]. Indeed, the unique property of the equation involved in the proof is the comparison principle, which both the fast diffusion equation and the p-Laplacian equation enjoy.

4.6.2 A lower bound for the finite extinction time

A first application of the Flux Lemma is a lower bound for the finite extinction time.

Lemma 4.5. *Under the assumptions of Lemma* 4.4 *and in the running notations, assuming moreover that $0 < R < 2R < R_0$, we have the following lower bound for the FET:*

$$T \geq \mathcal{K} R (R_0 - 2R)^{p-1} \left[\frac{1}{|B_{R_0}|} \int_{B_R} u_0(x)\, \mathrm{d}x \right]^{2-p}, \tag{4.67}$$

where $\mathcal{K}$ is a constant depending only on n and p. In particular, we obtain the lower bound for T in Theorem 4.4.

Proof. In order to derive this lower bound, we apply (4.63) to the annulus $A_0 := B_{2R} \setminus B_R$:

$$\int_{B_{2R}} u(x, s)\, \mathrm{d}x \leq \frac{k}{R} \int_s^T \int_{A_0} |\nabla u(x, \tau)|^{p-1}\, \mathrm{d}x\, d\tau, \tag{4.68}$$

We are going to use the following estimate for the gradient due to DiBenedetto and Herrero, cf. formula (0.8) in [53], that reads

$$
\begin{aligned}
\int_s^T \int_{B_{2R}} |\nabla u|^{p-1} \, \mathrm{d}x \, d\tau &\le \gamma(n,p) \left[1 + \frac{T-s}{\varepsilon^{2-p}(R_0-2R)^p} \right]^{\frac{p-1}{p}} \\
&\quad \times \int_s^T \int_{B_{R_0}} (T-\tau)^{\frac{1-p}{p}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, \mathrm{d}x \, d\tau \\
&\le \gamma(n,p) \left[1 + \frac{T-s}{\varepsilon^{2-p}(R_0-2R)^p} \right]^{\frac{p-1}{p}} (T-s)^{\frac{1-p}{p}} \\
&\quad \times \int_s^T \int_{B_{R_0}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, \mathrm{d}x \, d\tau,
\end{aligned}
\tag{4.69}
$$

when applied to any ball $B_{2R} \subset B_{R_0}$, for any $0 < s < T$ and for any $\varepsilon > 0$. The constant $\gamma(n,p)$ depends only on n and p. We join (4.68) and (4.69) and we let

$$
D(s) = \left(1 + \frac{T-s}{\varepsilon^{2-p}(R_0-2R)^p} \right)^{\frac{p-1}{p}},
$$

to obtain

$$
\int_{B_{2R}} u(x,s) \, \mathrm{d}x \le \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1-p}{p}} \int_s^T \int_{B_{R_0}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, \mathrm{d}x \, d\tau.
$$

Then there exists $\bar{s} \in (s,T)$ such that we have

$$
\begin{aligned}
\int_{B_{2R}} u(x,s) \, \mathrm{d}x &\le \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1-p}{p}} (T-s) \int_{B_{R_0}} \big(u(x,\bar{s})+\varepsilon\big)^{\frac{2(p-1)}{p}} \, \mathrm{d}x \\
&\le \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1}{p}} |B_{R_0}| \left[\frac{1}{|B_{R_0}|} \int_{B_{R_0}} \big(u(x,\bar{s})+\varepsilon\big) \, \mathrm{d}x \right]^{\frac{2(p-1)}{p}} \\
&= \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1}{p}} |B_{R_0}|^{\frac{2-p}{p}} \left[\int_{B_{R_0}} \big(u(x,\bar{s})+\varepsilon\big) \, \mathrm{d}x \right]^{\frac{2(p-1)}{p}}.
\end{aligned}
$$

where in the first step we have used the mean-value theorem for the time integral in the right-hand side, and in the second step the Hölder inequality. Using now the contractivity of the L^1 norm, we obtain

$$
\int_{B_{2R}} u(x,s) \, \mathrm{d}x \le \frac{k(n,p)}{R} D(s)(T-s)^{\frac{1}{p}} |B_{R_0}|^{\frac{2-p}{p}} \left[\int_{B_{R_0}} \big(u(x,s)+\varepsilon\big) \, \mathrm{d}x \right]^{\frac{2(p-1)}{p}}. \tag{4.70}
$$

We put now $s=0$. On the other hand, we take $\varepsilon > 0$ such that the following condition holds true:

$$
\varepsilon |B_{R_0}| = \int_{B_R} u_0(x) \, \mathrm{d}x.
$$

This condition implies that

$$D(0) \leq \frac{C(n,p)}{\varepsilon^{\frac{(2-p)(p-1)}{p}}(R_0-2R)^{p-1}} T^{\frac{p-1}{p}},$$

$$\int_{B_{R_0}} (u_0+\varepsilon)\ \mathrm{d}x = 2\int_{B_{R_0}} u_0(x)\ \mathrm{d}x = 2\int_{B_R} u_0(x)\ \mathrm{d}x,$$

the last equality being justified by the fact that $\mathrm{supp}(u_0) \subset B_R$. Coming back to (4.70), letting there $s=0$, replacing the precise value of ε and taking into account the previous remarks, we obtain:

$$\begin{aligned}\int_{B_R} u_0(x)\ \mathrm{d}x &\leq \frac{K(n,p)}{\varepsilon^{\frac{(2-p)(p-1)}{p}} R(R_0-2R)^{p-1}} T|B_{R_0}|^{\frac{2-p}{p}} \left(\int_{B_R} u_0(x)\ \mathrm{d}x\right)^{\frac{2(p-1)}{p}} \\ &\leq \frac{K(n,p)}{R(R_0-2R)^{p-1}} T|B_{R_0}|^{2-p} \left(\int_{B_R} u_0(x)\ \mathrm{d}x\right)^{p-1},\end{aligned}$$

where $K(n,p) = 2^{2(p-1)/p} C(n,p) k\gamma(n,p)$, k being the constant in (4.63). It follows that:

$$\left(\int_{B_R} u_0(x)\ \mathrm{d}x\right)^{2-p} \leq \frac{K(n,p)}{R(R_0-2R)^{p-1}} T|B_{R_0}|^{2-p},$$

hence the lower bound follows in the stated form, once we let $\mathcal{K} = K(n,p)$. □

4.6.3 Positivity for the minimal Dirichlet problem

The result of the Flux Lemma 4.4 can be interpreted as the transformation of the positivity information coming from the initial mass into positivity information in terms of energy. Our next goal is to transfer the positivity information for the energy obtained so far, to positivity for the solution itself in an annulus. To this end we will use again the above mentioned gradient estimate of [53], formula (0.8). We split the proof of the positivity estimate into several steps.

Step 1. Reversed space-time Sobolev inequalities along the flow. Let u be the solution of the MDP (4.61), in the assumption that $R_0 > 3R$. We begin by writing the estimate (4.63) in the ball of radius $7R/3$:

$$\int_{B_{7R/3}} u(x,s)\ \mathrm{d}s \leq \frac{k}{R}\int_s^T \int_{B_{8R/3}\setminus B_{7R/3}} |\nabla u(x,\tau)|^{p-1}\ \mathrm{d}x\, d\tau\,. \tag{4.71}$$

We now want to estimate the right-hand side in terms of a suitable mean value of u. The estimate we would like to have is quite uncommon, indeed it can be interpreted as a reversed Sobolev inequality on an annulus A_1, along the p-Laplacian flow. In general this kind of reversed inequalities tend to be false.

To this end, we cover the annulus $B_{8R/3} \setminus B_{7R/3}$ by smaller balls, of "good" radius, then we consider a covering with larger balls and we apply the estimate (4.69) for $|\nabla u|^{p-1}$. More precisely, we consider a family of balls $\{B_i\}_{i=1,N}$ with radius R_i, satisfying the following two conditions: that $B_{8R/3} \setminus B_{7R/3} \subset \bigcup_{i=1}^N B_i$ and that $R/6 < R_i < R/3$. For any ball B_i, we consider a larger, concentric ball B_i' with radius R_i', such that $R_i < R_i' < R/3$. From this construction, we deduce that

$$B_{8R/3} \setminus B_{7R/3} \subset \bigcup_{i=1}^{N} B_i \subset \bigcup_{i=1}^{N} B_i' \subset B_{3R} \setminus B_{2R} \subset B_{R_0} \setminus B_{2R},$$

which is useful, since we remain in a region where the Aleksandrov principle applies. We apply the estimate from [53] for any of the pairs (B_i, B_i') and we sum up to finally obtain the desired form for the reversed space-time Sobolev inequality:

$$\int_s^T \int_{B_{8R/3}\setminus B_{7R/3}} |\nabla u(x,\tau)|^{p-1}\, \mathrm{d}x\, d\tau \le \frac{N\gamma(n,p)}{(T-s)^{\frac{p-1}{p}}} D(s,\varepsilon) \int_s^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\, \mathrm{d}x\, d\tau, \tag{4.72}$$

Joining this with (4.71) we get

$$\int_{B_{7R/3}} u(x,s)\, \mathrm{d}s \le \frac{Nk(n)\gamma(n,p)}{R} D(s,\varepsilon)(T-s)^{\frac{1-p}{p}} \int_s^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\, \mathrm{d}x\, d\tau, \tag{4.73}$$

which holds for any $s \in [0,T]$ and $\varepsilon > 0$, where we have used the following notations

$$D(s,\varepsilon) := \left(1 + \frac{T-s}{\varepsilon^{2-p} K^p}\right)^{\frac{p-1}{p}}, \qquad K := \min_{i=1,N} (R_i' - R_i). \tag{4.74}$$

Remark. In the estimates above, the condition $B_{3R} \subset B_{R_0}$ can be replaced by $B_{2R+\varepsilon} \subset B_{R_0}$, for any $\varepsilon > 0$ fixed, with the same proof. That is why, the condition $R_0 > 2R$ is sufficient for the result to hold.

Step 2. Estimating time integrals. We are going to estimate the time integral in the right-hand side of (4.73) by splitting it in two parts. For any $0 \le s \le t \le T$ we have

$$\begin{aligned}
\int_s^t \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\, \mathrm{d}x &\le |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \int_s^t \left[\int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)\, \mathrm{d}x\right]^{\frac{2(p-1)}{p}} d\tau \\
&\le |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \int_s^t \left[\int_{B_{R_0}} (u+\varepsilon)\, \mathrm{d}x\right]^{\frac{2(p-1)}{p}} d\tau \\
&\le |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \int_s^t \left[\int_{B_{R_0}} u_0\, \mathrm{d}x + \varepsilon |B_{R_0}|\right]^{\frac{2(p-1)}{p}} d\tau \\
&= (t-s)|B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \left[\int_{B_R} u_0\, \mathrm{d}x + \varepsilon |B_{R_0}|\right]^{\frac{2(p-1)}{p}},
\end{aligned}$$

where we have used Hölder inequality in the first step, and then the $L^1(B_{R_0})$-contractivity for the MDP in the third step, while in the last step we take into account that $\operatorname{supp} u_0 \subset B_R$. We rescale ε in such a way that $\varepsilon = \alpha \int_{B_R} u_0 \, dx/|B_{R_0}|$, leaving $\alpha > 0$ as a free parameter that will be chosen later on. The final result of this step reads

$$\int_s^t \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, dx \, d\tau \le (1+\alpha)^{\frac{2(p-1)}{p}} (t-s)|B_{R_0} \setminus B_R|^{\frac{2-p}{p}} \left[\int_{B_R} u_0 \, dx\right]^{\frac{2(p-1)}{p}}. \quad (4.75)$$

Step 3. The critical time. Let us come back to (4.73) and put $s = 0$, so that

$$\begin{aligned}
\int_{B_R} u_0(x) \, dx &\le \frac{Nk(n)\gamma(n,p)}{R} D(0,\varepsilon) T^{\frac{1-p}{p}} \int_0^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, dx \, d\tau \\
&= \frac{Nk(n)\gamma(n,p)}{R} D(0,\varepsilon) T^{\frac{1-p}{p}} \left[\int_0^{t^*} \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, dx \, d\tau \right.\\
&\left. + \int_{t^*}^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, dx \, d\tau \right] \\
&\le \frac{Nk(n)\gamma(n,p)}{R} D(0,\varepsilon) T^{\frac{1-p}{p}} \left[(1+\alpha)^{\frac{2(p-1)}{p}} t^* |B_{R_0} \setminus B_{2R}|^{\frac{2-p}{p}} \left(\int_{B_R} u_0 \, dx\right)^{\frac{2(p-1)}{p}} \right.\\
&\left. + \int_{t^*}^T \int_{B_{R_0}\setminus B_R} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, dx \, d\tau \right],
\end{aligned}$$

where in the last step we have used (4.75) to estimate the first integral. Here t^* is a particular time that will be chosen later. We estimate now $D(0,\varepsilon)$, with our choice of ε, starting from the numeric inequality $(1+y)^{(p-1)/p} \le (2y)^{(p-1)/p} := \kappa\, y^{(p-1)/p}$, which holds for any $y > 1$,

$$D(0,\varepsilon) = \left(1 + \frac{T}{\varepsilon^{2-p}K^p}\right)^{\frac{p-1}{p}} \le \frac{\kappa T^{\frac{p-1}{p}}}{\alpha^{\frac{(2-p)(p-1)}{p}} K^{p-1}} \left[\frac{1}{|B_{R_0}|} \int_{B_R} u_0(x) \, dx\right]^{-\frac{(2-p)(p-1)}{p}},$$

where we have chosen $y = T/\big(\varepsilon^{2-p}K^p\big) > 1$. The condition, in terms of K (defined in (4.74)), becomes

$$K^p := \left[\min_{i=1,N} (R_i' - R_i)\right]^p < T\,\varepsilon^{p-2} = T \left[\alpha \int_{B_R} u_0 \frac{dx}{|B_{R_0}|}\right]^{p-2}. \quad (4.76)$$

We will check the compatibility of this condition after our choice of ε. Joining the above two estimates, we get

$$\begin{aligned}
\left(\int_{B_R} u_0 \, dx\right)^{1+\frac{(2-p)(p-1)}{p}} &\le \frac{k_0 |B_{R_0}|^{\frac{(2-p)(p-1)}{p}}}{RK^{p-1}\alpha^{\frac{(2-p)(p-1)}{p}}} \left[(1+\alpha)^{\frac{2(p-1)}{p}} t^* |B_{R_0}|^{\frac{2-p}{p}} \left(\int_{B_R} u_0 \, dx\right)^{\frac{2(p-1)}{p}} \right.\\
&\left. + \int_{t^*}^T \int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}} \, dx \, d\tau \right],
\end{aligned} \quad (4.77)$$

where we have used that $|B_{R_0} \setminus B_{2R}| < |B_{R_0}|$, and we have defined $k_0 := Nk(n)\gamma(n,p)\kappa$. We choose now the critical time t^* as

$$t^* = \frac{R}{2k_0}\left(\frac{K}{\alpha}\right)^{p-1}\left(\frac{\alpha}{1+\alpha}\right)^{\frac{2(p-1)}{p}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0\, \mathrm{d}x\right)^{2-p}. \tag{4.78}$$

It remains to check that $t^* \le T$, and this will be done after we fix the values of α and K.

Step 4. The mean-value theorem. First we substitute the value (4.78) of t^* in (4.77)

$$\left(\int_{B_R} u_0\, \mathrm{d}x\right)^{1+\frac{(2-p)(p-1)}{p}} \le \frac{2k_0\,|B_{R_0}|^{\frac{(2-p)(p-1)}{p}}}{RK^{p-1}\alpha^{\frac{(2-p)(p-1)}{p}}}\int_{t^*}^{T}\int_{B_{R_0}\setminus B_{2R}} (u+\varepsilon)^{\frac{2(p-1)}{p}}\, \mathrm{d}x\, d\tau,$$

then we apply the mean-value theorem to the time integral in the right-hand side and we obtain that there exists $t_1 \in [t^*, T]$ such that

$$\frac{RK^{p-1}}{2k_0(T-t^*)}\left(\int_{B_R} u_0\, \mathrm{d}x\right)^{1+\frac{(2-p)(p-1)}{p}} \le \left[\frac{|B_{R_0}|}{\alpha}\right]^{\frac{(2-p)(p-1)}{p}}\int_{B_{R_0}\setminus B_{2R}} \big(u(x,t_1)+\varepsilon\big)^{\frac{2(p-1)}{p}}\, \mathrm{d}x. \tag{4.79}$$

Step 5. Application of the Aleksandrov reflection principle. We are now in position to apply Proposition 4.3, in the form (4.66), to the right-hand side of the above estimate

$$\int_{B_{R_0}\setminus B_{2R}} \big(u(x,t_1)+\varepsilon\big)^{\frac{2(p-1)}{p}}\, \mathrm{d}x \le |B_{R_0}|\big(u(x_0,t_1)+\varepsilon\big)^{\frac{2(p-1)}{p}}, \tag{4.80}$$

note that the presence of ε does not affect the estimate. Joining (4.79) and (4.80), and recalling that we have rescaled $\varepsilon = \alpha\int_{B_R} u_0\, \mathrm{d}x/|B_{R_0}|$ we get

$$\left[u(x_0,t_1) + \frac{\alpha}{|B_{R_0}|}\int_{B_R} u_0\, \mathrm{d}x\right]^{\frac{2(p-1)}{p}} \ge \frac{\alpha^{\frac{(2-p)(p-1)}{p}} RK^{p-1}}{2k_0 T}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0\, \mathrm{d}x\right)^{1+\frac{(2-p)(p-1)}{p}},$$

or, equivalently,

$$u(x_0,t_1) \ge \alpha^{\frac{2-p}{2}}\left(\frac{RK^{p-1}}{2k_0T}\right)^{\frac{p}{2(p-1)}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0\, \mathrm{d}x\right)^{1+\frac{p(2-p)}{2(p-1)}} - \frac{\alpha}{|B_{R_0}|}\int_{B_R} u_0\, \mathrm{d}x = \mathcal{H}(\alpha),$$

which holds for any $\alpha > 0$. Immediately we see that $\mathcal{H}(0) = 0$ and in the limit $\alpha \to +\infty$ we get $\mathcal{H}(\alpha) \to -\infty$, since $1 < p < 2$. An optimization of $\mathcal{H}$ in α shows that it achieves its maximum value at the point

$$\overline{\alpha} = \left(\frac{2-p}{2}\right)^{\frac{2}{p}}\frac{K}{[2k_0]^{\frac{1}{p-1}}}\left(\frac{R}{T}\right)^{\frac{1}{p-1}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0(x)\, \mathrm{d}x\right)^{\frac{2-p}{p-1}}. \tag{4.81}$$

The value of the function $\mathcal{H}(\overline{\alpha})$ is strictly positive and takes the form

$$u(x_0,t_1) \ge \mathcal{H}(\overline{\alpha}) = \frac{p}{2-p}\left[\frac{2-p}{2}\right]^{\frac{2}{p}}\frac{K}{[2k_0]^{\frac{1}{p-1}}}\left[\frac{R}{T}\right]^{\frac{1}{p-1}}\left[\frac{1}{|B_{R_0}|}\int_{B_R} u_0\, \mathrm{d}x\right]^{\frac{1}{p-1}}, \tag{4.82}$$

which finally gives our first positivity estimate at the point t_1, once we check that all the choices of the parameters are compatible. Indeed, we first have to check the compatibility between (4.76) and (4.81), that is

$$K^2 := \left[\min_{i=1,N}\left\{R_i^{'} - R_i\right\}\right]^2 := \rho^2 R^2 < \frac{2^{\frac{2}{p}}(2k_0)^{\frac{1}{p-1}}}{(2-p)^{\frac{2}{p}}}\left[\frac{T}{R^{2-p}}\right]^{\frac{1}{p-1}}\left[\frac{1}{|B_{R_0}|}\int_{B_R} u_0 \, \mathrm{d}x\right]^{\frac{p-2}{p-1}}, \tag{4.83}$$

which is nothing but a restriction on the choice of the radii R_i and $R_i^{'}$ in terms of the data of the MDP, and allow to fix a value of ρ in terms of the data. It only remains to check that substituting the value $\overline{\alpha}$ in the expression (4.78) of t^*, we have $t^* \le T$, where T is the finite extinction time. From (4.78) and (4.81) we obtain

$$\begin{aligned} t^* &= \frac{R}{2k_0}\left(\frac{K}{\overline{\alpha}}\right)^{p-1}\left(\frac{\overline{\alpha}}{1+\overline{\alpha}}\right)^{\frac{2(p-1)}{p}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0 \, \mathrm{d}x\right)^{2-p} \\ &= \left[\frac{2\,\overline{\alpha}}{(2-p)(1+\overline{\alpha})}\right]^{\frac{2(p-1)}{p}} := kT\,, \end{aligned} \tag{4.84}$$

where $k \le 1$ if and only if

$$\overline{\alpha} = \left(\frac{2-p}{2}\right)^{\frac{2}{p}}\frac{K}{(2k_0)^{\frac{1}{p-1}}}\left(\frac{R}{T}\right)^{\frac{1}{p-1}}\left(\frac{1}{|B_{R_0}|}\int_{B_R} u_0(x) \, \mathrm{d}x\right)^{\frac{2-p}{p-1}} \le \frac{\left(\frac{2-p}{2}\right)^{\frac{2}{p}}}{1-\left(\frac{2-p}{2}\right)^{\frac{2}{p}}}, \tag{4.85}$$

and this condition is satisfied, since K is bounded as in (4.83), but the constant k_0 can be chosen arbitrarily large, since it comes from the upper bound (4.77).

Removing the dependence on T in the expression (4.85) *of* $\overline{\alpha}$. Let us note that formula (4.84) expresses t^* as an increasing function of $\overline{\alpha}$ whenever

$$\overline{\alpha} \le \frac{\left(\frac{2-p}{2}\right)^{\frac{2}{p}}}{1-\left(\frac{2-p}{2}\right)^{\frac{2}{p}}}.$$

Letting equality in the above expression we can remove T from the expression of $\overline{\alpha}$ and a posteriori we can conclude that t^* given by (4.84), does not depend on T. A convenient expression for t^* is given by

$$t^* = k^* R^{p-n(2-p)}\|u_0\|_{L^1(B_R(x_0))}^{2-p}, \tag{4.86}$$

where the constant k^* depends only on n, p.

Step 6. Positivity backward in time. In this step we recover positivity for any time $0 < t < t_1$, using an extension of the celebrated Benilan-Crandall estimates, cf. [22]. Indeed, the Benilan-Crandall estimate for the MDP reads

$$u_t(x,t) \le \frac{u(x,t)}{(2-p)t}, \tag{4.87}$$

hence the function $u(x,t)t^{-1/(2-p)}$ is non-increasing in time. It follows that for any time $t \in (0,t_1)$, we have:

$$u(x,t_1) \le t^{-\frac{1}{2-p}} t_1^{\frac{1}{2-p}} u(x,t) \le t^{-\frac{1}{2-p}} T^{\frac{1}{2-p}} u(x,t).$$

We join this last inequality with (4.82) and we obtain our main positivity result for solutions to MDP:

$$\left(\frac{p}{2-p}\right)^{p-1} \left(\frac{2-p}{2}\right)^{\frac{2(p-1)}{p}} \frac{\rho^{p-1}R^p}{2k_0 T} \frac{1}{|B_{R_0}|} \int_{B_R} u_0 \,\mathrm{d}x \le t^{-\frac{p-1}{2-p}} T^{\frac{p-1}{2-p}} u(x_0,t)^{p-1}. \tag{4.88}$$

We conclude by letting

$$k(n,p) = 2k_0\, \rho^{p-1}\, \frac{2-p}{p} \left(\frac{2}{2-p}\right)^{\frac{2}{p}}.$$

We thus proved the following positivity theorem for solutions to MDP.

Theorem 4.14. *Let $1 < p < 2$, let u be the solution to the Minimal Dirichlet Problem* (4.61) *and let T be its finite extinction time. Then $T > t^*$ and the following inequality holds true for any $t \in (0,t^*]$:*

$$u(x_0,t)^{p-1} \ge k(n,p) t^{\frac{p-1}{2-p}} T^{-\frac{1}{2-p}} \frac{R^p}{|B_{R_0}|} \int_{B_R} u_0 \,\mathrm{d}x. \tag{4.89}$$

In particular, the estimate (4.89) *establishes the positivity of u in the interior ball of the annulus up to the critical time t^* expressed by* (4.86).

4.6.4 Aronson-Caffarelli type estimates

We have obtained positivity estimates for initial times, namely $t \in (0,t^*)$ and now we want to see whether it is possible to extend such positivity estimates globally in time, i. e., for any $t \in (0,T)$. This can be done and leads to some kind of inequalities in the form of the celebrated Aronson-Caffarelli estimates valid for the degenerate/slow diffusions, cf. [6]. As a precedent two of the authors proved in [34] some kind of Aronson-Caffarelli estimates for the fast diffusion equation.

We begin by rewriting the positivity estimates in the form of the following alternative: either $t > t^*$, or

$$u(x_0,t)^{p-1} \ge k(n,p) t^{\frac{p-1}{2-p}} T^{-\frac{1}{2-p}} \frac{R^p}{|B_{R_0}|} \int_{B_R} u_0 \,\mathrm{d}x.$$

We recall now the expression of t^* given in (4.86)

$$t^* = k_* R^{p-n(2-p)} \|u_0\|_{L^1(B_R(x_0))}^{2-p}.$$

The above inequalities can be summarized in the following equivalent alternative: either

$$\frac{1}{|B_{R_0}|} \int_{B_R} u_0(x) \,\mathrm{d}x \le C_1(n,p) t^{\frac{1}{2-p}} R^{-\frac{p}{2-p}},$$

or

$$\frac{1}{|B_{R_0}|}\int_{B_R} u_0(x)\ \mathrm{d}x \le k(n,p)t^{-\frac{p-1}{2-p}}T^{\frac{1}{2-p}}\,R^{-p}u(x_0,t)^{p-1}.$$

Summing up the above estimates, we obtain, for any $t\in(0,T)$,

$$R^{-n}\int_{B_R} u_0(x)\ \mathrm{d}x \le C_1 t^{\frac{1}{2-p}}R^{-\frac{p}{2-p}} + C_2 t^{-\frac{p-1}{2-p}}T^{\frac{1}{2-p}}R^{-p}u(x_0,t)^{p-1}, \tag{4.90}$$

where C_1 and C_2 are constants depending only on n and p.

As already mentioned, the above Aronson-Caffarelli type estimates are global in time, but they provide quantitative lower bounds only for $0<t<t^*$. As far as we know, this kind of lower parabolic Harnack inequalities are new for the p-Laplacian.

Remark. Let us notice that, even working with initial data $u_0\in L^2(B_R)$, we never use the L^2 norm of the initial datum in a quantitative way, but only its L^1 norm. This observation allows for the approximation argument described in the next section.

4.7 Positivity for continuous local weak solutions

Throughout this section, u will be a non-negative and continuous local weak solution, cf. Definition 4.1, defined in $Q_T=\Omega\times(0,T)$, taking initial data $u_0\in L^1_{loc}(\Omega)$. We recall that $B_{R_0}(x_0)\subset\Omega$ and assume in all this section that $R_0>5R$, in order to compare u and the solution u_D of a suitable Minimal Dirichlet Problem. We never use the modulus of continuity of u.

4.7.1 Proof of Theorems 4.2 and 4.3

Fix a time $t\in(0,T_1)$ and a point $x_1\in\overline{B_R(x_0)}$, so that $B_R(x_0)\subset B_{2R}(x_1)\subset B_{(4+\varepsilon)R}(x_1)\subset B_{R_0}(x_0)$, for some $\varepsilon>0$ sufficiently small (more precisely, $\varepsilon>0$ should satisfy $R_0>(5+\varepsilon)R$). Since $u_0\chi_{B_R(x_0)}\in L^1(B_R(x_0))$, we can approximate it with functions $u_{0,j}\in L^2(B_R(x_0))$, such that $u_{0,j}\to u_0\chi_{B_R(x_0)}$ as $j\to\infty$ in the space $L^1(B_R(x_0))$. We consider now the following sequence of minimal Dirichlet problems in a ball centered at x_1:

$$\begin{cases} u_t=\Delta_p u, & \text{in } B_{(4+\varepsilon)R}(x_1)\times(0,T),\\ u(x,0)=u_{0,j}(x)\chi_{B_R(x_0)}(x), & \text{in } B_{(4+\varepsilon)R}(x_1),\\ u(x,t)=0, & \text{for } t>0 \text{ and } x\in\partial B_{(4+\varepsilon)R}(x_1),\end{cases}$$

which, by standard theory (see [52]), admits a unique continuous weak solution $u_{D,j}$, for which Theorem 4.14 applies. We then compare $u_{D,j}$ with the continuous solution to the problem $(\mathbb{D})$, which is our local weak solution u restricted to $B_{(4+\varepsilon)R}(x_1)\times(0,T)$. It follows that

$$u(x,t)\ge u_{D,j}(x,t), \qquad \text{and} \qquad T\ge T_{m,j},$$

where $T_{m,j}$ is the finite extinction time for $u_{D,j}$. We then apply Theorem 4.14 to $u_{D,j}$ to obtain

$$\begin{aligned} u_{D,j}(x_1,t)^{p-1} &\geq c\,R^p\,t^{\frac{p-1}{2-p}}T_{m,j}^{-\frac{1}{2-p}}\frac{1}{|B_{R_0}(x_1)|}\int_{B_{(4+\varepsilon)R}(x_1)} u_{0,j}(x)\chi_{B_R(x_0)}(x)\;\mathrm{d}x \\ &\geq c(n,p)R^{p-n}t^{\frac{p-1}{2-p}}T_{m,j}^{-\frac{1}{2-p}}\int_{B_R(x_0)} u_{0,j}(x)\;\mathrm{d}x, \end{aligned}$$

provided that $t < t_j^*$, with t_j^* as in the previous section (but applied to $u_{0,j}$). Taking into account that $u_{D,j}(x_1,t) \leq u(x_1,t)$ and that, in the previous estimates, t_j^* and $T_{m,j}$ depend only on the L^1 norm of $u_{0,j}$, we can pass to the limit in order to find that

$$u(x_1,t)^{p-1} \geq c(n,p)R^{p-n}t^{\frac{p-1}{2-p}}T_m^{-\frac{1}{2-p}}\int_{B_R(x_0)} u_0(x)\;\mathrm{d}x\,,$$

where $T_m = T_m(u_0) = \lim_{j\to\infty} T_{m,j}$, provided that $t < t^* = \lim_{j\to\infty} t_j^*$, as in the previous section. Moreover, t^* and T_m do not depend on the choice of the point $x_1 \in \overline{B_R(x_0)}$, but only on the support of the initial data which is fixed, we can take $x_1 = x_1(t)$ as the point where

$$u(x_1,t) = \inf_{x\in B_R(x_0)} u(x,t).$$

Thus, we arrive to the desired inequality (4.5). Moreover, by the same comparison we get the Aronson-Caffarelli type estimates (4.6) for any continuous local weak solution. □

Remark. The fact that $T(u) \geq T_m = T_m(u_0)$ for any continuous local weak solution u justifies the name of *minimal life time* that we give to T_m in the Introduction.

4.7.2 Proof of Theorem 4.4

Let $p_c < p < 2$. We divide the proof of Theorem 4.4 into several steps, following the lines of the similar result in [34].

Step 1. Scaling. Let u_R be the solution of the homogeneous Dirichlet problem in the ball $B_R(x_0)$, with initial datum $u_0 \in L^1(B_R)$ and with extinction time $T(u_0,R) < \infty$. Then the rescaled function

$$u(x,t) = \frac{M}{R^n}\,\overline{u}\left(\frac{x-x_0}{R}, \frac{t}{R^{np-2n+p}M^{2-p}}\right), \quad M = \int_{B_R} u_0\;\mathrm{d}x\,,$$

solves the homogeneous Dirichlet problem in $B(0,1)$, with initial datum $\overline{u}_0$ of mass 1 and with extinction time $\overline{T}$ such that $T(u_0,R) = R^{np-2n+p}M^{2-p}\overline{T}$. Therefore, we can work in the unit ball and with rescaled solutions.

Step 2. Barenblatt-type solutions. Consider the solution $\mathcal{B}$ of the homogeneous Dirichlet problem in the unit ball $B(0,1)$, with initial trace the Dirac mass, $\mathcal{B}(0) = \delta_0$. By comparison with the Barenblatt solutions of the Cauchy problem (that exist precisely for $p_c < p < 2$), we find that

$$\mathcal{B}(x,t) \leq C(n,p)t^{-n\vartheta_1}, \quad \text{for any } (x,t) \in B(0,1)\times[0,\infty).$$

By the concentration-comparison principle (see [131], [127]), it follows that the solution $\mathcal{B}$ extinguishes at the later time among all the solutions with initial datum of mass 1, call $T(\mathcal{B})$ its extinction time. We have to prove that $T(\mathcal{B}) < \infty$, that will be done by comparison with another solution, described below.

Step 3. Separate variable solution. Let us consider the solution

$$U_r(x,t) = (T_1 - t)^{\frac{1}{2-p}} X(x), \quad \text{in } B_r,\ r > 1,$$

with extinction time T_1 to be chosen later. Then, X is a solution of the elliptic equation $\Delta_p X + X/(2-p) = 0$ in B_{R_0}, hence it can be chosen radially symmetric and bounded from above and from below by the distance to the boundary. On the other hand, fix $t_0 > 0$ and let T_1 be given by $X(1)(T_1 - t_0)^{1/(2-p)} = C(p,n)t_0^{-n\vartheta_1}$.

Step 4. Comparison and end of proof. We compare the solutions $\mathcal{B}$ and U_r constructed above in the cylinder $Q_1 = B_1(0) \times [t_0, T_1)$. The comparison on the boundary is trivial and the initial data (at $t = t_0$) are ordered by the choice of t_0. It follows that $\mathcal{B}(x,t) \le U_r(x,t)$ in Q_1, hence their extinction times are ordered: $T(\mathcal{B}) \le T_1 < \infty$. Moreover, it is easy to check (by optimizing in t_0) that T_1 depends only on p and n, hence $\overline{T} \le T(\mathcal{B}) \le K(n,p)$, for any solution of the homogeneous Dirichlet problem in B_1 with extinction time $\overline{T}$. Coming back to the original variables, we find that

$$T(u_0, R) \le K(n,p) R^{np-2n+p} \|u_0\|_{L^1(B_R)}^{2-p},$$

which is the upper bound of Theorem 4.4. The lower bound has been obtained in Subsection 4.6.2. The lower Harnack inequality (4.8) follows immediately from estimate (4.5). □

4.7.3 Upper bounds for the extinction time and proof of Theorem 4.5

In this subsection we prove universal upper estimates for the finite extinction time T, in the range $1 < p < p_c$, in terms of suitable norms of the initial datum u_0, and we subsequently prove Theorem 4.5. Throughout this subsection, u is a solution to a global homogeneous Dirichlet or Cauchy problem in $\Omega \subseteq \mathbb{R}^n$, with initial datum u_0, whose regularity will be treated below.

Bounds in terms of the L^{r_c} norm. Following the ideas of Benilan and Crandall [23], we begin by differentiating in time the global L^r norm of the solution $u(t)$ to a global (Cauchy or Dirichlet) problem:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^r \,\mathrm{d}x &= -r(r-1) \int_\Omega u^{r-2} |\nabla u|^p \,\mathrm{d}x = -\frac{r(r-1)p^p}{(r+p-2)^p} \int_\Omega \left| \nabla u^{\frac{r+p-2}{p}} \right|^p \,\mathrm{d}x \\ &\le -\frac{r(r-1)p^p \mathcal{S}_p^p}{(r+p-2)^p} \left[\int_\Omega u^{\frac{(r+p-2)p^*}{p}} \,\mathrm{d}x \right]^{\frac{p}{p^*}}, \end{aligned} \tag{4.91}$$

where in the last step we used the Sobolev inequality; here, $p^* = np/(n-p)$ and $\mathcal{S}_p$ is the Sobolev constant. Note that $(r+p-2)p^*/p = r$ if and only if $r = r_c$. If $p > p_c$, then $r_c < 1$, hence the global L^{r_c} norm increases, originating a *Backward Effect* (see [127]).

We thus restrict ourselves to $p < p_c$, in which case the constant $r_c(r_c-1)p^p/(r_c+p-2)^p$ is positive. Then, (4.91) implies the following closed differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{r_c}^{r_c} \leq -\frac{r_c(r_c-1)p^p\mathcal{S}_p^p}{(r_c+p-2)^p}\|u(t)\|_{r_c}^{\frac{pr_c}{p^*}},$$

whose integration leads to

$$\|u(t)\|_{r_c}^{2-p} \leq \|u(s)\|_{r_c}^{2-p} - K(t-s), \qquad K = \frac{r_c(r_c-1)p^{p+1}\mathcal{S}_p^p}{n(r+p-2)^p}, \tag{4.92}$$

which holds for any $0 \leq s \leq t \leq T$ and for any $p < p_c$. Letting now $s = 0$ and $t = T$ in (4.92), we obtain the following universal upper bound for the extinction time:

$$T \leq K^{-1}\|u_0\|_{r_c}^{2-p}. \tag{4.93}$$

In particular, if the initial datum $u_0 \in L^{r_c}(\Omega)$, then the solution u extinguishes in finite time.

Bounds in terms of other L^r norms. As we have seen, the condition $u_0 \in L^{r_c}(\Omega)$ does not allow for the Local Smoothing Effect to hold. That is why, in this part we obtain upper bounds for the extinction time T in terms of other global L^r norms, with the expected condition $r > r_c$, but only in bounded domains Ω. Following ideas from [32] and [34], we consider a function $f \in W_0^{1,p}(\Omega)$, and we apply the Poincaré, Sobolev and Hölder inequalities as follows:

$$\|f\|_q \leq \|f\|_p^{\vartheta}\|f\|_{p^*}^{1-\vartheta} \leq \mathcal{P}_\Omega^{\vartheta}\mathcal{S}_p^{1-\vartheta}\|\nabla f\|_p, \tag{4.94}$$

for any $q \in (p, p^*)$, where $\vartheta \in (0,1)$, $\mathcal{P}_\Omega$ is the Poincaré constant of the domain Ω and $\mathcal{S}_p$ is the Sobolev constant. We let in (4.94)

$$f = u^{\frac{r+p-2}{p}}, \qquad q = \frac{pr}{r+p-2}, \qquad \vartheta = \frac{r-r_c}{r},$$

which are in the range where this inequality applies, since $q > p$ for any $p < 2$ and $q < p^*$ if and only if $r > r_c$. We then restrict ourselves to the case $r > r_c$ and, replacing in (4.94), we obtain

$$\|u\|_r^{\frac{r+p-2}{rp}} \leq \mathcal{P}_\Omega^{1-\frac{r_c}{r}}\mathcal{S}_p^{\frac{r_c}{r}}\left\|\nabla u^{\frac{r+p-2}{p}}\right\|_p. \tag{4.95}$$

We elevate (4.95) at power p and join it then with the inequality (4.91) for the derivative of the global L^r norm. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_r^r = -\frac{r(r-1)p^p}{(r+p-2)^p}\left\|\nabla u^{\frac{r+p-2}{p}}\right\|_p^p \leq K_0\|u(t)\|_r^{\frac{r+p-2}{r}},$$

where

$$K_0 := \frac{r(r-1)p^p\,\mathcal{S}_p^{\frac{r}{p\,r_c}}\mathcal{P}_\Omega^{\frac{p(r-r_c)}{r}}}{(r+p-2)^p}.$$

By integration over $[s,t] \subseteq [0,T]$, we obtain that

$$\|u(t)\|_r^{2-p} \leq \|u(s)\|_r^{2-p} - K_0(t-s), \tag{4.96}$$

for any $0 \le s \le t \le T$ and for any $r > r_c$. We let now $s = 0$, $t = T$ in (4.96) and we obtain an upper bound for the extinction time:

$$T^{\frac{1}{2-p}} \le K_0^{-\frac{1}{2-p}} \|u_0\|_r = \left[\frac{r(r-1)p^p \, \mathcal{S}_p^{\frac{r}{p\, r_c}} \mathcal{P}_\Omega^{\frac{p(r-r_c)}{r}}}{(r+p-2)^p} \right]^{-\frac{1}{2-p}} \|u_0\|_r = c_1 R^{-\frac{rp+n(p-2)}{r(2-p)}} \|u_0\|_r \,, \tag{4.97}$$

since the Poincaré constant $\mathcal{P}_\Omega \sim R$ and where c_1 only depends on p, r, n and goes to zero as $r \to 1$. In particular, any solution u of a homogeneous Dirichlet problem in Ω, with $u_0 \in L^r$, $r > r_c$, extinguishes in finite time.

Remarks. (i) The above results prove that a global Sobolev and Poincaré inequality implies that the solution extinguishes in finite time and gives quantitative upper bounds for the extinction time T.

(ii) Direct applications of these bounds in the estimates (4.5) and (4.6) prove Theorem 4.5.

4.8 Harnack inequalities

By joining the local lower and upper bounds obtained in the previous sections, we obtain various forms of Harnack inequalities. These are expressions relating the maximum and minimum of a solution inside certain parabolic cylinders. In the well known linear case one has

$$\sup_{Q_1} u(x,t) \le C \inf_{Q_2} u(x,t). \tag{4.98}$$

The main idea is that the formula applies for a large class of solutions and the constant C that enters the relation does not depend on the particular solution, but only on the data like p, n and the size of the cylinder. The cylinders in the standard case are supposed to be ordered, $Q_1 = B_{R_1}(x_0) \times [t_1, t_2]$, $Q_2 = B_{R_2}(x_0) \times [t_3, t_4]$, with $t_1 \le t_2 < t_3 \le t_4$ and $R_1 < R_2$.

It is well-known that in the degenerate nonlinear elliptic or parabolic problems a plain form of the inequality does not hold. In the work of DiBenedetto and collaborators, see the book [52] or the recent work [55], versions are obtained where some information of the solution is used to define so-called intrinsic sizes, like the size of the parabolic cylinder(s), that usually depends on $u(x_0, t_0)$. They are called *intrinsic Harnack inequalities.*

The Harnack Inequalities of [52, 55], in the supercritical range then read: *There exist positive constants $\overline{c}$ and $\overline{\delta}$ depending only on p, n, such that for all $(x_0, t_0) \in \Omega \times (0,T)$ and all cylinders of the type*

$$B_R(x_0) \times \left(t_0 - c\, u(x_0,t_0)^{2-p}(8R)^p, t_0 + c\, u(x_0,t_0)^{2-p}(8R)^p\right) \subset \Omega \times (0,T)\,, \tag{4.99}$$

we have

$$\overline{c}\, u(x_0,t_0) \le \inf_{x \in B_R(x_0)} u(x,t)\,,$$

for all times $t_0 - \overline{\delta}\, u(x_0,t_0)^{2-p} R^p < t < t_0 + \overline{\delta}\, u(x_0,t_0)^{2-p} R^p$. The constants $\overline{\delta}$ and $\overline{c}$ tend to zero as $p \to 2$ or as $p \to p_c$.

They also give a counter-example in the lower range $p < p_c$, by producing an explicit local solution that does not satisfy any kind of Harnack inequality (neither of the types called intrinsic, elliptic, forward, backward) if one fixes "a priori" the constant c. At this point a natural question is posed:

What form may take the Harnack estimate, if any, when p is in the subcritical range $1 < p \le p_c$?
We will give an answer to this question.

If one wants to apply the above result to a local weak solution defined on $\Omega \times [0,T]$, where T is possibly the extinction time, one should care about the size of the intrinsic cylinder, namely the intrinsic hypothesis (4.99) reads

$$\overline{c}\, u(x_0,t_0) \le \left[\frac{\min\{t_0, T-t_0\}}{(8R)^p}\right]^{\frac{1}{2-p}} \qquad \text{and} \qquad \mathrm{dist}(x_0,\partial\Omega) < \frac{R}{8}\,, \tag{4.100}$$

This hypothesis is guaranteed in the good range by the fact that solutions with initial data in $\mathrm{L}^1_{\mathrm{loc}}$ are bounded, while in the very fast diffusion range hypothesis (4.100) fails, and should be replaced by :

$$u(x_0,t) \le \frac{c_{p,n}}{\varepsilon^{\frac{2r\vartheta_r}{2-p}}} \left[\frac{\|u(t_0)\|_{\mathrm{L}^r(B_R)}\, R^d}{\|u(t_0)\|_{\mathrm{L}^1(B_R)} R^{\frac{n}{r}}}\right]^{2r\vartheta_r} \left[\frac{t_0}{R^p}\right]^{\frac{1}{2-p}}\,.$$

This local upper bound can be derived by our local smoothing effect of Theorem 4.1, whenever $t_0 + \varepsilon t^*(t_0) < t < t_0 + t^*(t_0)$, where the critical time is defined by a translation in formula (4.86) as follows

$$t^*(t_0) = k^* R^{p-n(p-2)} \|u(t_0)\|^{2-p}_{L^1(B_R(x_0))}\,, \tag{4.101}$$

full details are given below, in the proof of Theorem 4.6. In this new intrinsic geometry we obtain the plain form of intrinsic Harnack inequalities of Theorem 4.6, namely

There exist constants $h_1\,, h_2$ depending only on $d,p,r,$ such that, for any $\varepsilon \in [0,1]$ the following inequality holds

$$\inf_{x\in B_R(x_0)} u(x, t\pm\theta) \ge h_1 \varepsilon^{\frac{rp\vartheta_r}{2-p}} \left[\frac{\|u(t_0)\|_{L^1(B_R)} R^{\frac{n}{r}}}{\|u(t_0)\|_{L^r(B_R)} R^n}\right]^{rp\vartheta_r + \frac{1}{2-p}} u(x_0,t),$$

for any $t_0 + \varepsilon t^*(t_0) < t \pm \theta < t_0 + t^*(t_0)$.

We have obtained various forms of Harnack inequalities, namely

Forward Harnack inequalities. These inequalities compare the supremum at a time t_0 with the infimum of the solution at a later time $t_0 + \vartheta$. These kind of Harnack inequalities hold for the linear heat equation as well: we recover the classical result just by letting $p \to 2$.

Elliptic-type Harnack inequalities. These inequalities are typical of the fast diffusion range, indeed they compare the infimum and the supremum of the solution at the same time, namely consider $\theta = 0$ above. It is false for the Heat equation and for the degenerate p-Laplacian, as one can easily check by plugging respectively the gaussian heat kernel or the Barenblatt solutions. This kind of inequalities are true for the fast diffusion processes, as noticed by two of the authors in [34, 35] and by DiBenedetto et al. in [55, 57] in the supercritical range.

Backward Harnack inequalities. These inequalities compare the supremum at a time t_0 with the infimum of the solution at a previous time $t_0 - \vartheta$. This backward inequality is a typical feature of the fast diffusion processes, that somehow takes into account the phenomena of extinction in finite time, as already mentioned in Subsection 2.4.

In the very fast diffusion range $1 < p \le p_c$ our intrinsic Harnack inequality represents the first and only known result. In the good range, $p_c < p < 1$ we can take $r = 1$, so that the ratio of L^r norms simplifies and we recover the result of [52, 55] with a different proof.

Throughout this section T_m will denote the finite extinction time for the minimal Dirichlet problem (4.61), i. e. the so-called minimal life time of any continuous local weak solution.

4.8.1 Intrinsic Harnack inequalities. Proof of Theorem 4.6

Let u be a nonnegative, continuous local weak solution of the fast p-Laplacian equation in a cylinder $Q = \Omega \times (0,T)$, with $1 < p < 2$, taking an initial datum $u_0 \in L^r_{\rm loc}(\Omega)$, with $r \ge \max\{1, r_c\}$. Let $x_0 \in \Omega$ be a fixed point, such that $\operatorname{dist}(x_0, \partial\Omega) > 5R$. We recall the notation T_m for the minimal life time associated to the initial data u_0 and the ball $B_R(x_0)$, and we denote the critical time

$$t^*(s) = k^* R^{p-n(p-2)} \|u(s)\|^{2-p}_{L^1(B_R(x_0))}, \quad t^* = t^*(0),$$

which is a shift in time of the expression (4.86).

With these notations and assumptions, we first prove a generalized form of the Harnack inequality, that holds for initial times, or equivalently for small intrinsic cylinders, and in which we allow the constants to depend also on T_m.

Theorem 4.15. *For any $t_0 \in (0, t^*]$, and any $\theta \in [0, t_0/2]$ such that $t_0 + \theta \le t^*$, the following forward/backward/elliptic Harnack inequality holds true:*

$$\inf_{x \in B_R(x_0)} u(x, t_0 \pm \theta) \ge H u(x_0, t_0), \tag{4.102}$$

where

$$H = C R^{\frac{np-2n+p}{(p-1)(2-p)}} \left[\frac{\|u_0\|_{L^1(B_R)}}{T_m^{\frac{1}{2-p}}} \right]^{\frac{1}{p-1}} \left[R^{\frac{p}{2-p}} \frac{\|u_0\|^{rp\vartheta_r}_{L^r(B_R)}}{t_0^{\frac{rp\vartheta_r}{2-p}}} + 1 \right]^{-1}.$$

and C depends only on r, p, n. H goes to 0 as $t_0 \to 0$.

Proof. Let us recall first that, from Theorem 4.2, $u(x_0, t_0) > 0$ for $t_0 < t^*$. Let us fix $t_0 \in (0, t^*)$ and choose $\theta > 0$ sufficiently small such that $t_0 + \theta \le t^*$ and $t_0 \pm \theta \ge t_0/3$. We plug these quantities into the lower estimate (4.2) to get:

$$\begin{aligned}
\inf_{x \in B_R} u(x, t_0 \pm \theta) &\ge C (t_0 \pm \theta)^{\frac{1}{2-p}} R^{\frac{p-n}{p-1}} T_m^{-\frac{1}{(2-p)(p-1)}} \|u_0\|^{\frac{1}{p-1}}_{L^1(B_R)} \\
&\ge C \left(\frac{t_0}{3 R_0^p} \right)^{\frac{1}{2-p}} R^{\frac{np-2n+p}{(p-1)(2-p)}} T_m^{-\frac{1}{(2-p)(p-1)}} \|u_0\|^{\frac{1}{p-1}}_{L^1(B_R)}.
\end{aligned}$$

On the other hand, we use the local upper bound (4.5), in the following way:

$$u(x_0,t_0) \le C_3 \left[R^{\frac{p}{2-p}} \frac{\|u_0\|_{L^r(B_{2R})}^{rp\vartheta_r}}{t_0^{\frac{rp\vartheta_r}{2-p}}} + 1 \right] \left(\frac{t_0}{R^p} \right)^{\frac{1}{2-p}}.$$

Joining the two previous estimates, we obtain the desired form of the inequality. □

We are now ready to prove Theorem 4.6, which is our main intrinsic Harnack inequality.
Proof of Theorem 4.6 We may assume that $t_0 = 0$, hence $t^*(t_0) = t^*$; the general result follows by translation in time. We use again the local smoothing effect of Theorem 4.1 as before and we estimate:

$$\begin{aligned} u(x_0,t_0) &\le C_3 \left[1 + \frac{\|u_0\|_{L^r(B_R)}^{rp\vartheta_r}}{t_0^{\frac{rp\vartheta_r}{2-p}}} R^{\frac{p}{2-p}} \right] \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}} \le C_4 \left[\frac{\|u_0\|_{L^r(B_R)}^{rp\vartheta_r}}{(\varepsilon t^*)^{\frac{rp\vartheta_r}{2-p}}} R^{\frac{p}{2-p}} \right] \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}} \\ &\le \frac{C_5}{\varepsilon^{\frac{rp\vartheta_r}{2-p}}} \left[\frac{\|u_0\|_{L^r(B_R)} R^n}{\|u_0\|_{L^1(B_R)} R^{\frac{n}{r}}} \right]^{rp\vartheta_r} \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}}, \end{aligned} \tag{4.103}$$

where the second step in the inequality above follows from the assumption that $t_0 \ge \varepsilon t^*$. On the other hand, we can remove the dependence on T_m in the lower estimate of Theorem 4.15, using the results in Subsection 4.7.3, namely:

$$T_m^{\frac{1}{2-p}} \le C(r,p,n) R^{\frac{p}{2-p} - \frac{n}{r}} \|u_0\|_{L^r(B_R)}, \quad r \ge \max\{1, r_c\},$$

hence the lower estimate becomes

$$\inf_{x \in B_R(x_0)} u(x, t \pm \theta) \ge C_6 \left[\frac{\|u_0\|_{L^1(B_R)} R^{\frac{n}{r}}}{\|u_0\|_{L^r(B_R)} R^n} \right]^{-\frac{1}{p-1}} \left[\frac{t_0}{R^2} \right]^{\frac{1}{2-p}}. \tag{4.104}$$

Joining the inequalities (4.103) and (4.104), we obtain the estimate (4.11) as stated. We pass from $[0, t^*]$ to any interval $[t_0, t_0 + t^*(t_0)]$ by translation in time. □
Alternative form of the Harnack inequality. The following alternative form of the Harnack inequality is given avoiding the intrinsic geometry and the waiting time $\varepsilon \in [0,1]$. An analogous version, for the degenerate diffusion of p-Laplacian type, can be found in [56].

Theorem 4.16. *Under the running assumptions, there exists C_1, $C_2 > 0$, depending only on r, n, p, such that the following inequality holds true:*

$$\sup_{x \in B_R} u(x,t) \le C_1 \frac{\|u(t_0)\|_{L^r(B_{2R})}^{rp\vartheta_r}}{t^{n\vartheta_r}} + C_2 \left[\frac{\|u(t_0)\|_{L^r(B_R)} R^n}{\|u(t_0)\|_{L^1(B_R)} R^{\frac{n}{r}}} \right]^{\frac{1}{p-1}} \inf_{x \in B_R} u(x, t \pm \theta), \tag{4.105}$$

for any $0 \le t_0 < t \pm \theta < t_0 + t^*(t_0) < T$.

The proof is very easy and it consists only in joining the upper estimate (4.4) with the lower estimate (4.104) above. We leave the details to the interested reader.

Remark. In the good fast-diffusion range $p > p_c$, we can let $r = 1$ and obtain

$$\sup_{x\in B_R} u(x,t) \le C_1 \frac{\|u(t_0)\|_{L^1(B_{2R})}^{p\vartheta_1}}{t^{n\vartheta_1}} + C_2 \inf_{x\in B_R} u(x, t\pm\theta).$$

4.9 Special Energy Inequality. Rigorous Proof of Theorem 4.7

We devote this section to the proof of Theorem 4.7, and to further generalizations and applications of it. Throughout this section, by admissible test function we mean $\varphi \in C_c^2(\Omega)$ as specified in the statement of Theorem 4.7.

We have presented in the Introduction the basic, formal calculation leading to inequality (4.12). Our task here will be to give a detailed justification of this formal proof. To this end we state and prove in full detail an auxiliary result.

Proposition 4.4. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a strictly positive smooth function, let $\varphi \ge 0$ be a nonnegative admissible test function. Define the associated Φ-Laplacian operator*

$$\Delta_\Phi u := \operatorname{div}\left[\Phi'\left(|\nabla u|^2\right)\nabla u\right]. \tag{4.106}$$

Then the following inequality holds true for continuous weak solutions to the Φ-Laplacian evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi\left(|\nabla u|^2\right)\varphi\,\mathrm{d}x + \frac{2}{n}\int_\Omega (\Delta_\Phi u)^2\varphi\,\mathrm{d}x \le \int_\Omega \left[\Phi'\left(|\nabla u|^2\right)\right]^2\left(|\nabla u|^2\right)\Delta\varphi\,\mathrm{d}x. \tag{4.107}$$

Remark. Let us remark that the p-Laplacian is obtained by taking $\Phi(w) = \frac{2}{p}w^{p/2}$, but we stress the fact that this choice of Φ falls out the smoothness requirement of the above proposition.

Proof. This proof is a straightforward generalization of the above formal proof of Theorem 4.7. Denote $w = |\nabla u|^2$. Take a test function $\varphi \ge 0$ as in the assumptions. We perform a time derivation of the energy associated to the Φ-Laplacian

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(w)\varphi\,\mathrm{d}x &= -2\int_\Omega \operatorname{div}\big[\Phi'(w)(\nabla u)\,\varphi\big]\Delta_\Phi u\,\mathrm{d}x\\ &= -2\int_\Omega (\Delta_\Phi u)^2\varphi\,\mathrm{d}x - 2\int_\Omega (\Delta_\Phi u)\Phi'(w)(\nabla u\cdot\nabla\varphi)\,\mathrm{d}x.\end{aligned}$$

We then apply identity (4.14) and inequality (4.15) for the vector field $F = \Phi(w)|\nabla u|$ and finally obtain (4.107). □

The rest of the argument is based on suitable approximations of the p-Laplacian equation by the Φ-Laplacians introduced above; it will be divided into several steps.

STEP 1. APPROXIMATING PROBLEMS. We now let $\Phi_\varepsilon(w) = \frac{2}{p}(w+\varepsilon^2)^{p/2}$, which is our approximation for the p-Laplacian nonlinearity. We also consider a fixed sub-cylinder $Q' \subset Q_T$ of the form $Q' = B_R \times (T_1, T_2)$ where $B_R \subset \Omega$ is a small ball and $0 < T_1 < T_2 < T$. Choose moreover T_1 such that $\|\nabla u(T_1)\|_{L^p(B_R)} = K < \infty$, which is true for a. e. time
We introduce the following approximating Dirichlet problem in Q':

$$(P_\varepsilon)\begin{cases} u_{\varepsilon,t} = \Delta_{\Phi_\varepsilon} u_\varepsilon := \operatorname{div}\left[\left(|\nabla u_\varepsilon|^2+\varepsilon^2\right)^{(p-2)/2}\nabla u_\varepsilon\right], \text{ in } Q', \\ u_\varepsilon(x,T_1) = u(x,T_1), \text{ for any } x \in B_R, \\ u_\varepsilon(x,t) = u(x,t), \text{ for } x \in \partial B_R,\ t \in (T_1,T_2). \end{cases} \tag{4.108}$$

Since the equation in this problem is neither degenerate, nor singular, and the boundary data are continuous by our assumptions, the solution u_ε of (P_ε) is unique and belongs to $C^\infty(Q')$ (see [98] for the standard parabolic theory), hence the result of Proposition 4.4 holds true for u_ε. Moreover, u_ε satisfies the following weak formulation:

$$\begin{aligned} &\int_{B_R} u_\varepsilon(x,t_2)\varphi(x,t_2)\,\mathrm{d}x - \int_{B_R} u_\varepsilon(x,t_1)\varphi(x,t_1)\,\mathrm{d}x \\ &\qquad + \int_{t_1}^{t_2}\int_{B_R}\left[-u_\varepsilon(x,s)\varphi_t(x,s) + \left(|\nabla u_\varepsilon|^2+\varepsilon^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon(x,s)\cdot\nabla\varphi(x,s)\right]\mathrm{d}x\,\mathrm{d}s = 0, \end{aligned} \tag{4.109}$$

for any times $T_1 \le t_1 < t_2 \le T_2$ and for any test function $\varphi \in W^{1,2}\big(T_1,T_2;L^2(B_R)\big) \cap L^p\big(T_1,T_2;W_0^{1,p}(B_R)\big)$. Conversely, if a function $v \in C^\infty(Q')$ satisfies the weak formulation (4.109) and takes as boundary values u in the continuous sense, then by uniqueness of the Dirichlet problem, we can conclude $v = u_\varepsilon$.

STEP 2: UNIFORM LOCAL ENERGY ESTIMATES FOR u_ε. In the next steps, we are going to establish uniform estimates (i. e. independent of ε) for some suitable norms of the solution u_ε to (P_ε). In the first part, we deal with the local L^p norm of the gradient of the solution. Starting from (4.107), we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x &\le \frac{p}{2}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\Delta\varphi\,\mathrm{d}x \\ &\le \frac{p}{2}\left[\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x\right]^{\frac{2(p-1)}{p}}\left[\int_{B_R}\varphi^{-\frac{2(p-1)}{2-p}}(\Delta\varphi)^{\frac{p}{2-p}}\,\mathrm{d}x\right]^{\frac{2-p}{p}} \\ &= C(\varphi)\left[\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\,\mathrm{d}x\right]^{\frac{2(p-1)}{p}}, \end{aligned}$$

where in the last inequality we applied Hölder inequality with the exponents $p/(2-p)$ and $p/2(p-1)$, and we have set

$$C(\varphi) = \frac{p}{2}\left[\int_{B_R}\varphi^{-\frac{2(p-1)}{2-p}}(\Delta\varphi)^{\frac{p}{2-p}}\,\mathrm{d}x\right]^{\frac{2-p}{p}}. \tag{4.110}$$

We assume for the moment that $C(\varphi) < \infty$. We then arrive to the following closed differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_\varepsilon(t) \le C(\varphi) Y_\varepsilon(t)^{\frac{2(p-1)}{p}},$$

where

$$Y_\varepsilon(t) = \int_{B_R} \left(\varepsilon^2 + |\nabla u_\varepsilon(x,t)|^2\right)^{\frac{p}{2}} \varphi(x)\, \mathrm{d}x.$$

An integration over (t_0, t_1) gives

$$Y_\varepsilon(t_1)^{\frac{2-p}{p}} - Y_\varepsilon(t_0)^{\frac{2-p}{p}} \le C(\varphi)(t_1 - t_0),$$

for any $T_1 \le t_0 < t_1 \le T_2$. Letting $t_0 = T_1$ and observing that $t := t_1 - t_0 < T$, we find:

$$\left[\int_{B_R} \left(\varepsilon^2 + |\nabla u_\varepsilon(t)|^2\right)^{\frac{p}{2}} \varphi\, \mathrm{d}x\right]^{\frac{2-p}{p}} \le C(\varphi)T + \left[|B_R| + \|\nabla u(T_1)\|^p_{L^p(B_R)}\right]^{\frac{2-p}{p}},$$

where in the last step we have used the numerical inequality $(a+b)^{p/2} \le a^{p/2} + b^{p/2}$, valid for any a, $b > 0$ and $p < 2$. On the other hand, we see that

$$\int_{B_R} |\nabla u_\varepsilon|^p \varphi\, \mathrm{d}x \le \int_{B_R} \left(\varepsilon^2 + |\nabla u_\varepsilon|^2\right)^{\frac{p}{2}} \varphi\, \mathrm{d}x \le \left[|B_R| + \|\nabla u(T_1)\|^p_{L^p(B_R)}\right]^{\frac{2-p}{p}} + C(\varphi)T\,. \tag{4.111}$$

From the choice of T_1 such that $\|\nabla u(T_1)\|_{L^p(B_R)} < \infty$, it follows that the right-hand side is uniformly bounded. Hence the family $\{|\nabla u_\varepsilon|\}$ has a uniform bound in $L^\infty([T_1, T_2]; L^p_{loc}(B_R))$, which does not depend on ε. The choice of φ such that $C(\varphi) < \infty$ follows from Lemma 4.6, part (b), applied for $\beta = p/(2-p)$.

Finally, from standard results in measure theory we know that the set of times $t \in (0, T)$ such that $\|\nabla u(t)\|_{L^p(B_R)} < \infty$ is a dense set. Hence, for any $t_0 \in (0, T)$ given, there exists $T_1 < t_0$ with the above property, and, consequently, a generic parabolic cylinder $B_R \times [t_0, T_2]$ can be considered as part of a bigger cylinder $B_R \times [T_1, T_2]$ with T_1 as above, for which our approximation process applies.

Step 3. A uniform Hölder estimate for $\{u_\varepsilon\}$. We prove that the family $\{u_\varepsilon\}$ admits a uniform Hölder regularity up to the boundary. We will use Theorem 1.2, Chapter 4 of [52], and to this end we change the notations to $a(x,t,u,\nabla u) = \left(|\nabla u|^2 + \varepsilon^2\right)^{\frac{p-2}{2}} \nabla u$ and we prove the following inequalities:

(a) Since $(2-p)/2 < 1$, we have that $\left(|\nabla u|^2 + \varepsilon^2\right)^{\frac{2-p}{2}} \le |\nabla u|^{2-p} + \varepsilon^{2-p}$, and

$$a(x,t,u,\nabla u) \cdot \nabla u = \frac{|\nabla u|^2}{(\varepsilon^2 + |\nabla u|^2)^{(2-p)/2}} \ge \frac{|\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}}.$$

In order to apply the above mentioned result of [52], we have to find a constant $C_0 > 0$ and a nonnegative function φ_0 such that

$$\frac{|\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}} \ge C_0 |\nabla u|^p - \varphi_0(x,t),$$

or equivalently

$$\varphi_0(x,t) \geq \frac{C_0\varepsilon^{2-p}|\nabla u|^p - (1-C_0)|\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}} = \frac{1}{2}\frac{\varepsilon^{2-p}|\nabla u|^p - |\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}},$$

by taking $C_0 = 1/2$. If $|\nabla u| \geq \varepsilon$, then the right-hand side in the previous inequality is nonpositive and the existence of φ_0 is trivial. If $|\nabla u| < \varepsilon$, we can write:

$$\begin{aligned}\frac{\varepsilon^{2-p}|\nabla u|^p - |\nabla u|^2}{\varepsilon^{2-p} + |\nabla u|^{2-p}} &\leq \frac{\varepsilon^{2-p}|\nabla u|^p - |\nabla u|^2}{2|\nabla u|^{2-p}} = \frac{\varepsilon^{2-p} - |\nabla u|^{2-p}}{2|\nabla u|^{2(1-p)}} \\ &= \frac{1}{2}\left(\varepsilon^{2-p} - |\nabla u|^{2-p}\right)|\nabla u|^{2(p-1)} \leq \frac{\varepsilon^p}{2},\end{aligned}$$

hence we can take $\varphi_0 \equiv 1$.

(b) Since $p-2<0$, it follows that $\left(|\nabla u|^2 + \varepsilon^2\right)^{(p-2)/2} \leq |\nabla u|^{p-2}$, hence

$$\left|a(x,t,u,\nabla u)\right| = \left(|\nabla u|^2 + \varepsilon^2\right)^{(p-2)/2}|\nabla u| \leq |\nabla u|^{p-1}.$$

Joining the inequalities in (a) and (b) and taking into account that u is Hölder continuous (cf. [57] and Appendix A2), the family of Dirichlet problems (P_ε) that we consider satisfies the assumptions of Theorem 1.2, Chapter 4 of [52] in an uniform way, independent on ε, since the boundary and initial data are Hölder continuous with the same exponent as u. We conclude then that the family $\{u_\varepsilon\}$ is uniformly Hölder continuous up to the boundary in $\overline{Q'}$. By the Arzelà-Ascoli Theorem, we obtain that, eventually passing to a subsequence, $u_\varepsilon \to \tilde{u}$ uniformly in $\overline{Q'}$.

Step 4. Passing to the limit in (P_ε). The strategy will be the following: we pass to the limit $\varepsilon \to 0$ in the weak formulation (4.109) for (P_ε), in order to get the local weak formulation (4.2) for the original problem. We can pass to the limit in the terms without gradients using the uniform convergence proved in the previous step. On the other hand, we recall that $\{|\nabla u_\varepsilon| : \varepsilon > 0\}$ is uniformly bounded in $L^\infty\big(T_1,T_2;L^p_{loc}(B_R)\big)$, by Step 2, then, up to subsequences, there exists v such that, $\nabla u_\varepsilon \to v$ weakly in $L^q\big(T_1,T_2;L^p_{loc}(B_R)\big)$, for any $1 \leq q < +\infty$. Next, we can identify $v = \nabla\tilde{u}$, which gives that $u_\varepsilon \to \tilde{u}$ in $L^\infty\big(T_1,T_2;W^{1,p}_{loc}(B_R)\big)$. From this, we can pass to the limit also in the term containing gradients in the local weak formulation of (P_ε).

From the uniform convergence in $\overline{Q'}$ (cf. Step 3) and the considerations above, we deduce that the limit $\tilde{u}$ is actually a continuous weak solution of the following Dirichlet problem

$$(DP)\left\{\begin{array}{l} v_t = \Delta_p v, \text{ in } Q', \\ v(x,T_1) = u(x,T_1), \text{ for any } x \in B_R, \\ v(x,t) = u(x,t), \text{ for } x \in \partial B_R,\ t \in (T_1,T_2). \end{array}\right. \tag{4.112}$$

On the other hand, the continuous local weak solution u is a solution of the same Dirichlet problem. By comparison (that holds, since both solutions are continuous up to the boundary), it follows that $u = \tilde{u}$. We have thus proved that our approximation converges to the continuous solutions of the p-Laplacian equation.

STEP 5: CONVERGENCE IN MEASURE OF THE GRADIENTS. In this step, we will improve the convergence of ∇u_ε to ∇u. More precisely, we prove that the gradients converge in measure, which is stronger than the weak L^p convergence established in the previous steps. We follow ideas from the paper [21], having as starting point the following inequality for vectors a, $b \in \mathbb{R}^n$

$$(a-b)\cdot(|a|^{p-2}a-|b|^{p-2}b) \geq c_p \frac{|a-b|^2}{|a|^{2-p}+|b|^{2-p}}, \tag{4.113}$$

for some $c_p > 0$ for all $1 < p < 2$. This inequality is proved in Appendix A3 with optimal constant $c_p = \min\{1, 2(p-1)\}$. To prove the convergence in measure, take $\lambda > 0$ and decompose as in [21]

$$\begin{aligned}\{|\nabla u_{\varepsilon_1} - \nabla u_{\varepsilon_2}| > \lambda\} \subset & \Big\{ \{|\nabla u_{\varepsilon_1}| > A\} \cup \{|\nabla u_{\varepsilon_2}| > A\} \cup \{|u_{\varepsilon_1} - u_{\varepsilon_2}| > B\}\Big\} \\ & \cup \Big\{|\nabla u_{\varepsilon_1}| \leq A,\ |\nabla u_{\varepsilon_2}| \leq A,\ |\nabla u_{\varepsilon_1} - \nabla u_{\varepsilon_2}| > \lambda,\ |u_{\varepsilon_1} - u_{\varepsilon_2}| \leq B\Big\} \\ & := S_1 \cup S_2,\end{aligned}$$

for any ε_1, $\varepsilon_2 > 0$ and for any $A > 0$, $B > 0$ and $\lambda > 0$; we will choose A and B later. Since $\{\nabla u_\varepsilon : \varepsilon > 0\}$ is uniformly bounded in $L^p(B_R)$, for t fixed, and that $\{u_\varepsilon\}$ is Cauchy in the uniform norm, for any $\delta > 0$ given, we can choose $A = A(\delta) > 0$ sufficiently large and $B = B(\delta) > 0$ such that $|S_1| < \delta$. On the other hand, in order to estimate $|S_2|$, we observe that

$$S_2 \subset \Big\{|u_{\varepsilon_1} - u_{\varepsilon_2}| \leq B,\ (\nabla u_{\varepsilon_1} - \nabla u_{\varepsilon_2})\cdot(|\nabla u_{\varepsilon_1}|^{p-2}\nabla u_{\varepsilon_1} - |\nabla u_{\varepsilon_2}|^{p-2}\nabla u_{\varepsilon_2}) \geq \frac{C\lambda^2}{2A^{2-p}}\Big\},$$

where we have used the definition of S_2 and the inequality (4.113). Letting $\mu = C\lambda^2/2A^{2-p}$ and estimating further, we obtain

$$\begin{aligned}|S_2| &\leq \frac{1}{\mu} \iint\limits_{\{|u_{\varepsilon_1} - u_{\varepsilon_2}| \leq B\}} (\nabla u_{\varepsilon_1} - \nabla u_{\varepsilon_2})\cdot(|\nabla u_{\varepsilon_1}|^{p-2}\nabla u_{\varepsilon_1} - |\nabla u_{\varepsilon_2}|^{p-2}\nabla u_{\varepsilon_2})\, \mathrm{d}x\, \mathrm{d}t \\ &\leq \frac{1}{\mu}\int_{T_1}^{T_2}\int_{B_R} (u_{\varepsilon_1} - u_{\varepsilon_2})(\Delta_p u_{\varepsilon_1} - \Delta_p u_{\varepsilon_2})\, \mathrm{d}x\, \mathrm{d}t,\end{aligned}$$

where the integration by parts does not give boundary integrals, since $u_{\varepsilon_1} = u_{\varepsilon_2} = u$ on the parabolic boundary of the cylinder Q'. From the previous steps, we can replace $\Delta_p u_{\varepsilon_i}$ by $\Delta_{\Phi_{\varepsilon_i}} u_{\varepsilon_i} = \partial_t u_{\varepsilon_i}$, $i = 1, 2$ without losing too much (less than $\delta/3$ for ε_1, ε_2 sufficiently small), and the last estimate becomes

$$|S_2| \leq \frac{1}{2\mu}\int_{B_R}\int_{T_1}^{T_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(u_{\varepsilon_1} - u_{\varepsilon_2})^2\right]\, \mathrm{d}x\, \mathrm{d}t + \frac{2\delta}{3} \leq \delta,$$

for μ sufficiently large (or, equivalently, for $\lambda > 0$ sufficiently large) and for ε_1, $\varepsilon_2 < \varepsilon = \varepsilon(\delta)$ sufficiently small. This proves that for any $\delta > 0$, there exist $\lambda = \lambda(\delta) > 0$ and $\varepsilon = \varepsilon(\delta) > 0$ such that

$$\big|\{|\nabla u_{\varepsilon_1}| - |\nabla u_{\varepsilon_2}| > \lambda\}\big| \leq \delta, \quad \forall\, \varepsilon_1, \varepsilon_2 < \varepsilon(\delta),\ \lambda > \lambda(\delta),$$

that is, the family $\{\nabla u_\varepsilon\}$ is Cauchy in measure, hence convergent in measure. The limit coincides with the already established weak limit, which is ∇u.

STEP 6: PASSING TO THE LIMIT IN THE INEQUALITY. We have already proved that the weak solution u_ε of (P_ε) satisfies the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\ \mathrm{d}x+\frac{p}{n}\int_{B_R}(\Delta_{\Phi_\varepsilon}u)^2\varphi\ \mathrm{d}x\le\frac{p}{2}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\Delta\varphi\ \mathrm{d}x,\quad(4.114)$$

where $\Phi_\varepsilon(w)=\frac{2}{p}\left(w+\varepsilon^2\right)^{\frac{p}{2}}$. ¿From the previous step we know that $\nabla u_\varepsilon\to\nabla u$ in measure, hence, by passing to a suitable subsequence if necessary, the convergence is also true a. e. in Q'. From this fact, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{\frac{p}{2}}\varphi\ \mathrm{d}x\to\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R}|\nabla u|^p\varphi\ \mathrm{d}x\,,\qquad(4.115)$$

as $\varepsilon\to 0$, in distributional sense in $\mathcal{D}'(T_1,T_2)$, for any suitable test function φ. On the other hand, the continuous embedding $L^p(B_R)\subset L^{2(p-1)}(B_R)$, valid since $2(p-1)<p$ whenever $1<p<2$, implies

$$\int_{B_R}|\nabla u_\varepsilon|^{2(p-1)}\varphi\ \mathrm{d}x\le C\int_{B_R}|\nabla u_\varepsilon|^p\varphi\ \mathrm{d}x,$$

with a positive constant C independent of u, and for any suitable test function φ. We can easily see that the sequence $|\nabla u_\varepsilon|$ is weakly convergent in $\mathrm{L}^p(B_R)$, since

$$\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\mathrm{d}x\le\int_{B_R}\left(1+|\nabla u_\varepsilon|^{2(p-1)}\right)\mathrm{d}x\le C\left(\int_{B_R}1+|\nabla u_\varepsilon|^p\right)\mathrm{d}x\le K<+\infty\,,\qquad(4.116)$$

where in the last step we have used inequality (4.111) of Step 2, and K does not depend on $\varepsilon>0$. It is a well known fact that if a sequence is uniformly bounded in $\mathrm{L}^p(B_R)$ and converges in measure, then it converges strongly in any $\mathrm{L}^q(B_R)$, for any $1\le q<p$, and in particular for $q=2(p-1)<p$, whenever $p<2$. The same holds for $\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}$, by inequality (4.116). Summing up, we have proved that

$$\int_{B_R}\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{p-1}\Delta\varphi\ \mathrm{d}x\to\int_{B_R}|\nabla u|^{2(p-1)}\Delta\varphi\ \mathrm{d}x\,.\qquad(4.117)$$

It remains to analyze the second term in (4.114), which is bounded as a difference of the other two terms, and this implies that $u_{\varepsilon,t}$ is uniformly bounded in $L^2([T_1,T_2];L^2_{loc}(B_R))$. Up to subsequences, there exists $v\in L^2(T_1,T_2;L^2_{loc}(B_R))$ such that $u_{\varepsilon,t}\to v$ weakly in $L^2(T_1,T_2;L^2_{loc}(B_R))$ and we can identify easily $v=u_t$. Using the weak lower semicontinuity of the (local) L^2 norm, we obtain:

$$\liminf_{\varepsilon\to 0}\int_{B_R}(\Delta_{\Phi_\varepsilon}u)^2\varphi\ \mathrm{d}x=\liminf_{\varepsilon\to 0}\int_{B_R}u^2_{\varepsilon,t}\varphi\ \mathrm{d}x\ge\int_{B_R}u_t^2\varphi\ \mathrm{d}x=\int_{B_R}(\Delta_p u)^2\varphi\ \mathrm{d}x,\quad(4.118)$$

that finally implies inequality (4.12) for the solution u in B_R. Since the ball B_R and the time interval $[T_1,T_2]$ were arbitrarily chosen, we obtain (4.12) as in the statement of the theorem. □

Remarks. (i) From (4.12), we deduce directly that $u_t \in L^2(0,T;L^2_{loc}(\Omega))$, which is an improvement with respect to the L^1_{loc} regularity.
(ii) A closer inspection of the proof reveals that with minor modifications we can prove the inequality (4.107) of Proposition 4.4 also for general nonnegative Φ, thus allowing degeneracies and singularities of the corresponding Φ-Laplacian equation. More precisely, let us consider nonnegative functions Φ satisfying the following inequalities:

$$\Phi'(|\nabla u|^2)|\nabla u|^2 \geq C_0|\nabla u|^p - \psi_0(x,t),$$

and

$$|\Phi'(|\nabla u|^2)||\nabla u| \geq C_1|\nabla u|^{p-1} + \psi_1(x,t),$$

where C_0, $C_1 > 0$ and ψ_0, ψ_1 are nonnegative functions such that $\psi_0 \in L^s(0,T;L^q(\Omega))$ and $\psi_1^{p/(p-1)} \in L^s(0,T;L^q(\Omega))$, where $1 \leq s,q \leq \infty$ and

$$\frac{1}{s} + \frac{n}{p\,q} < 1.$$

These technical hypothesis appear in DiBenedetto's book [52].

4.9.1 Local upper bounds for the energy

In this subsection we derive local upper energy estimates, as an application of Theorem 4.7.

Theorem 4.17. *Let u be a continuous local weak solution of the fast p-Laplacian equation, with $1 < p < 2$, as in Definition 4.1, corresponding to an initial datum $u_0 \in L^r_{\rm loc}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is any open domain containing the ball $B_{R_0}(x_0)$. Then, for any $0 \leq s \leq t$, and any $0 < R < R_0$ and any $x_0 \in \Omega$ such that $B_{R_0}(x_0) \subset \Omega$, the following inequality holds true:*

$$\left[\int_{B_R(x_0)} |\nabla u(x,t)|^p \,\mathrm{d}x\right]^{(2-p)/p} \leq \left[\int_{B_{R_0}(x_0)} |\nabla u(x,s)|^p \,\mathrm{d}x\right]^{(2-p)/p} + K(t-s), \qquad (4.119)$$

where the positive constant K has the form

$$K = \frac{C_{p,n}}{(R_0-R)^2}|B_{R_0}\setminus B_R|^{(2-p)/p}, \qquad (4.120)$$

and where $C_{p,n}$ is a positive constant depending only on p and n.

Proof. We begin with inequality (4.12) and we drop the first term in the right-hand side, which is nonpositive:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p\varphi \,\mathrm{d}x \leq \frac{p}{2}\int_\Omega |\nabla u|^{2(p-1)}\Delta\varphi \,\mathrm{d}x.$$

An application of Hölder inequality, with conjugate exponents $p/2(p-1)$ and $p/(2-p)$, leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p\varphi \,\mathrm{d}x \leq C(\varphi)\left[\int_\Omega |\nabla u|^p\varphi \,\mathrm{d}x\right]^{2(p-1)/p}, \qquad (4.121)$$

where

$$C(\varphi) = \frac{p}{2}\left[\int_\Omega |\Delta\varphi|^{\frac{p}{2-p}}\varphi^{-\frac{2(p-1)}{2-p}}\,\mathrm{d}x\right]^{(2-p)/p} < +\infty\,,$$

since has the same expression as in (4.110). An integration over (s,t) gives

$$\left[\int_\Omega |\nabla u(x,t)|^p\varphi(x)\,\mathrm{d}x\right]^{(2-p)/p} \le \left[\int_\Omega |\nabla u(x,s)|^p\varphi(x)\,\mathrm{d}x\right]^{(2-p)/p} + \frac{(2-p)}{p}C(\varphi)(t-s).$$

We conclude by observing that the constant $C(\varphi)$ is exactly the same as (4.110) and thus we can repeat the same observation made there to express it in the desired form. $\square$

Remarks. (i) It is essential in the above inequality that $p<2$, since the constant explodes in the limit $p\to 2$. Indeed such kind of estimates are false for the heat equation, that is for $p=2$.
(ii) The constant also explodes when $R/R_0 \to 1$. Indeed,

$$K \sim C\frac{(R_0^n - R^n)^{(2-p)/p}}{(R_0-R)^2} \sim C(R_0-R)^{(2-3p)/p}.$$

4.9.2 Lower bounds for the $\mathrm{L}^1_{\mathrm{loc}}-$norm

In this subsection, we establish local lower bounds for the mass, in the following form:

Theorem 4.18. *Let u be a local weak solution of the fast p-Laplacian equation, with $1<p<2$, as in Definition 4.1, corresponding to an initial datum u_0, $|\nabla u_0|^p \in L^1_{loc}(\Omega)$, where $\Omega\subseteq\mathbb{R}^n$ is any open domain containing the ball $B_{R_0}(x_0)$. Then, for any $0\le t\le s$ and for any $0<R<R_0$, the following inequality holds true:*

$$\int_{B_R(x_0)} u(x,t)\,\mathrm{d}x \le \int_{B_{R_0}(x_0)} u(x,s)\,\mathrm{d}x + C\left[\left(\int_\Omega |\nabla u(x,s)|^p\varphi(x)\,\mathrm{d}x\right)^{1/p} + K^{\frac{1}{2-p}}\,|t-s|^{\frac{1}{2-p}}\right], \tag{4.122}$$

where

$$C = \overline{C}_{p,n}(R_0-R)|B_{R_0}\setminus B_R|^{\frac{p-1}{p}}, \qquad K = \frac{C_{p,n}}{(R_0-R)^2}|B_{R_0}\setminus B_R|^{(2-p)/p}, \tag{4.123}$$

with $\overline{C}_{p,n}$ and $C_{p,n}$ depending only on p and n.

Remarks. (i) The limits as $R\to+\infty$ give mass conservation for the Cauchy problem, when $p_c<p<2$, while in the subcritical range $1<p<p_c$ it indicates how much mass is lost at infinity. This estimates are new as far as we know.
(ii) The estimate (4.122) is different from that of Theorem 4.10, since it applies for $0\le t\le s$, and provides a local lower bound for the mass. Moreover, it allows us to obtain lower bounds

for the finite extinction time T, in the cases it occurs, just by letting $s = T$ and $t = 0$ in (4.122), as follows:

$$C^{p-2}\, K^{-1} \left[\int_{B_R(x_0)} u_0(x) \right]^{2-p} \, \mathrm{d}x \le T\,. \tag{4.124}$$

Proof. We begin by performing a time derivative of the local mass

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)\varphi(x)\, \mathrm{d}x &= \int_\Omega \mathrm{div}\left(|\nabla u(x,t)|^{p-2}\nabla u(x,t)\right)\varphi(x)\, \mathrm{d}x \\ &\ge -\int_\Omega |\nabla u(x,t)|^{p-1}|\nabla\varphi(x)|\, \mathrm{d}x. \end{aligned}$$

We then estimate the right-hand side by applying Hölder inequality with exponents $(p-1)/p$ and $1/p$, then using Lemma 4.6 with $\alpha = p$. Integrating over the time interval (t,s), we obtain the desired estimate. We leave the straightforward details to the reader. □

4.10 Panorama, open problems and existing literature

We recall here the values of $p_c = 2n/(n+1)$ and of the critical line $r_c = \max\{n(2-p)/p\,,\, 1\}$.

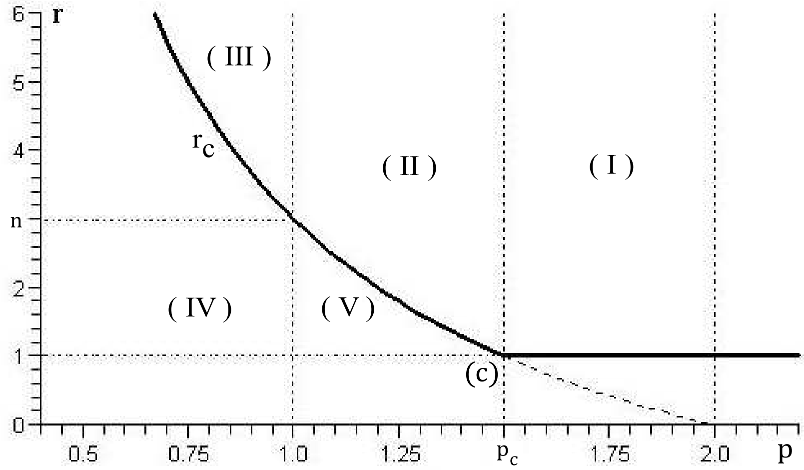

Figure 4.1: The critical line for the Smoothing Effect

(I) *Good Fast Diffusion Range:* $p \in (p_c, 2)$ and $r \ge 1$. In this range the local smoothing effect holds, cf. Theorem 4.1, as well as the positivity estimates of Theorem 4.2 and the Aronson-Caffarelli type estimates of Theorem 4.3. The intrinsic forward/backward/elliptic Harnack inequality Theorem 4.6 holds in this range. This is the only range in which there are some other works on Harnack inequalities. Indeed in the pioneering work of DiBenedetto and Kwong [54] there appeared for the first time the intrinsic Harnack inequalities for fast diffusion processes related to the p-Laplacian, now classified as forward Harnack inequalities. See also [57] for an excellent survey on these topics. In a recent paper DiBenedetto, Gianazza and Vespri [55] improve on the previous work by proving elliptic, forward and backward Harnack inequalities for more general operators of p-Laplacian type, but always in this "good range".

(II) *Very Fast Diffusion Range:* $p \in (1, p_c)$ and $r > r_c > 1$. In this range the local smoothing effect holds, cf. Theorem 4.1, as well as the positivity estimates of Theorem 4.2 and the Aronson-Caffarelli type estimates of Theorem 4.3. The intrinsic forward/backward/elliptic Harnack inequality Theorem 4.6 holds in this range as well, showing that if one allow the constants to depend on the initial data, then the form of Harnack inequalities is the same. No other kind of positivity, smoothing or Harnack estimates are known in this range, and our results represent a breakthrough in the theory of the singular p-Laplacian, indeed in [55] there is an explicit counterexample that shows that Harnack inequalities of backward, forward or elliptic type, are not true in general in this range, if the constants depend only on p and n.

The open question is now: If one wants absolute constants, what is the relation between the supremum and the infimum, if any?

(c) *Critical Case:* $p = p_c$ and $r > r_c = 1$. The local upper and lower estimates of zone (II) apply, as well as the Harnack inequalities. As previously remarked, all of our results are stable and consistent when $p = p_c$.

(III) and (IV) *Very Singular Range:* $0 < p \le 1$ with $r > r_c$ or $0 < p \le 1$ with $r < r_c$. In the range $p < 1$ the multidimensional p-Laplacian formula does not produce a parabolic equation. A theory in one dimension has been started in [14, 115], while radial self-similar solutions in several dimensions are classified in [79]. For reference to $p = 1$, the so-called total variation flow, cf. [3, 19].

(V) *Very Fast Diffusion Range:* $1 < p < p_c$ and $r \in [1, r_c]$. It is well known that the smoothing effect is not true in general, since initial data are not in L^p with $p > p_c$, cf. [127]. Lower estimates are as in (II). In general, Harnack inequalities are not possible in this range since solution may no be (neither locally) bounded.

4.10.1 Some general remarks

- We stress the fact that our results are completely local, and they apply to any kind of initial-boundary value problem, in any Euclidean domain: Dirichlet, Neumann, Cauchy, or problem for large solutions, namely when $u = +\infty$ on the boundary, etc. Natural extensions are fast diffusion problems for more general p-Laplacian operators and fast diffusion problems on manifolds.

- We calculate (almost) explicitly all the constants, through all the chapter.

- We have not entered either into the derivation of Hölder continuity and further regularity from the Harnack inequalities. This is a subject extensively treated in the works of DiBenedetto et al., see [57, 52, 55] and references therein.

- Summing up, no other results but ours are known in the lower range $p \le p_c$, and essentially one is known in the good range, and it refers to a different point of view.

- A combination of the techniques developed in this chapter and in [34], allow to extend the local smoothing effects, or the positivity estimates as well as the intrinsic Harnack inequalities to the *doubly nonlinear equation*

$$\partial_t u = \Delta_p u^m,$$

for which the fast diffusion range is understood as the set of exponents $m > 0$ and $p > 1$ such that $m(p-1) \in (0,1)$. Basic existence, uniqueness and regularity results on this equation, that allow for extensions of our results, appear in [58] and in [82]. We will not enter into the analysis of the extension in the present study.

Appendix

A1. Choice of particular test functions

In this appendix we show how we choose special test functions φ in various steps of the proof of our Local Smoothing Effect. We express these technical results in the form of the following

Lemma 4.6. *(a) For any open set $\Omega \subset \mathbb{R}^n$, for any two balls $B_R \subset B_{R_0} \subset \Omega$, and for any $\alpha > 0$, there exists a test function $\varphi \in C_c^\infty(\Omega)$ such that*

$$0 \le \varphi \le 1, \qquad \varphi \equiv 1 \text{ in } B_R, \qquad \varphi \equiv 0 \text{ outside } B_{R_0}\,, \tag{4.125}$$

and

$$\int_\Omega |\nabla\varphi|^\alpha \varphi^{1-\alpha}\,\mathrm{d}x < \frac{C_1}{(R_0-R)^\alpha}|A| < \infty, \tag{4.126}$$

where $C_1(n,\alpha)$ is a positive constant and $A = B_{R_0} \setminus B_R$.
(b) In the same conditions as in part (a), for any $\beta > 0$, there exists a test function $\varphi \in C_c^\infty(\Omega)$ satisfying (4.125) and such that

$$\int_\Omega |\Delta\varphi|^\beta \varphi^{1-\beta}\,\mathrm{d}x < \frac{C_2}{(R_0-R)^{2\beta}}|A| < \infty,$$

where $C_2(n,\beta)$ is a positive constant.

Proof. Let ψ be a radially symmetric C_c^∞ function which satisfies (4.125). It is easy to find ψ (see also [34]) satisfying the following estimates:

$$|\nabla\psi(x)| \le \frac{K_1}{(R_0-R)}, \qquad |\Delta\psi(x)| \le \frac{K_2}{(R_0-R)^2},$$

where K_1 and K_2 are positive constants depending only on n. Take $\varphi = \psi^\gamma$, where $\gamma > 0$ will be chosen later. It is clear that φ satisfies (4.125). We calculate:

$$|\nabla\varphi| = \gamma\psi^{\gamma-1}|\nabla\psi|, \qquad \Delta\varphi = \gamma\psi^{\gamma-1}\Delta\psi + \gamma(\gamma-1)\psi^{\gamma-2}|\nabla\psi|^2.$$

In order to prove part (a), we take $\gamma \geq \max\{1, \alpha\}$ and we remark that $\nabla\varphi$ is supported in the annulus A to estimate:

$$\int\limits_\Omega |\nabla\varphi|^\alpha \varphi^{1-\alpha}\,\mathrm{d}x \leq \gamma^\alpha \int\limits_A \psi^{\gamma-\alpha}\frac{K_1^\alpha}{(R_0-R)^\alpha}\,\mathrm{d}x < C_1(n)\frac{(K_1\gamma)^\alpha}{(R_0-R)^\alpha}|A|\,.$$

In order to prove part (b), we estimate:

$$|\Delta\varphi|^\beta\varphi^{1-\beta} \leq c\,[\gamma(\gamma-1)]^\beta\,\psi^{(\gamma-2)\beta+\gamma(1-\beta)}(|\Delta\psi| + |\nabla\psi|^2)^\beta.$$

Thus, choosing $\gamma > \max\{1, 2\beta\}$ and taking into account that $\Delta\varphi$ is supported in the annulus A, we obtain

$$\int\limits_\Omega |\Delta\varphi|^\beta\varphi^{1-\beta}\,\mathrm{d}x \leq \frac{C_2}{(R_0-R)^{2\beta}}|A|\,,$$

where $C_2 = C_2(p, n, \beta, \gamma)$ is a positive constant. $\square$

A2. Boundedness, regularity and local comparison

Let us recall now some well known regularity results for local weak solutions as introduced in Definition 4.1, given in Theorem 2.25 of [57]:

Theorem. *If u is a bounded local weak solution of (4.1) in Q_T, then u is locally Hölder continuous in Q_T. More precisely, there exist constants $\alpha \in (0,1)$ and $\gamma > 0$ such that, for every compact subset $K \subset Q_T$, and for every points (x_1, t_1), $(x_2, t_2) \in K$, we have:*

$$\big|u(x_1,t_1) - u(x_2,t_2)\big| \leq \gamma\|u\|_{L^\infty(Q_T)}\left[\frac{|x_1-x_2| + \|u\|_{L^\infty(Q_T)}^{\frac{p-2}{p}}|t_1-t_2|^{\frac{1}{p}}}{\mathrm{dist}(K, \partial Q_T)}\right]^\alpha,$$

where

$$\mathrm{dist}(K, \partial Q_T) = \inf_{(x,t)\in K, (y,s)\in\partial\Omega}\left\{|x-y|, \|u\|_{L^\infty(Q_T)}^{(p-2)/p}|t-s|^{1/p}\right\}\,,$$

and by ∂Q_T we understand the parabolic boundary of Q_T. The constants α and γ depend only on n and p.

Remark. The above theorem holds whenever u is a locally bounded function of space and time. We have used this result just in some technical steps: we begin with bounded local strong solution, which thanks to the above result are Hölder continuous. By the way, we can prove the smoothing effect for any local strong solution, independently of this continuity result, we thus obtain a posteriori that any local strong solution is Hölder continuous.

A3. A useful inequality related to the p-Laplacian

We prove the following inequality, used in some technical steps of the proof of Theorem 4.7.

Lemma 4.7. *For any vectors $a, b \in \mathbb{R}^n$, and for $1 < p \leq 2$, we have:*

$$(a-b)\cdot(|a|^{p-2}a - |b|^{p-2}b) \geq c_p \frac{|a-b|^2}{|a|^{2-p}+|b|^{2-p}}, \tag{4.127}$$

where the optimal constant is achieved when $a\cdot b = |a||b|$ and is given by $c_p = \min\{1, 2(p-1)\}$, if $1 < p < 2$, and $c_2 = 2$.

Proof. When $p = 2$, the inequality becomes a trivial equality with $c_2 = 2$. We next assume that $1 < p < 2$ and we rewrite inequality (4.127) as follows

$$\left(|a|^{2-p}+|b|^{2-p}\right)\left[|a|^p + |b|^p - \left(|a|^{p-2}+|b|^{p-2}\right) a\cdot b\right] \geq c_p\left(|a|^2+|b|^2 - 2a\cdot b\right)$$

or, equivalently, in the form

$$(1-c_p)\left(|a|^2+|b|^2-2a\cdot b\right) + |a|^{2-p}|b|^p + |a|^p|b|^{2-p} - \left(\frac{|a|^{2-p}}{|b|^{2-p}} + \frac{|b|^{2-p}}{|a|^{2-p}}\right) a\cdot b \geq 0$$

that can be reduced to

$$\left(\frac{|a|^{2-p}}{|b|^{2-p}} + \frac{|b|^{2-p}}{|a|^{2-p}} + 2(1-c_p)\right) a\cdot b \leq |a|^{2-p}|b|^p + |a|^p|b|^{2-p} + (1-c_p)\left(|a|^2+|b|^2\right).$$

Now it is clear that the worst case occurs when $a\cdot b = |a||b|$, since we always have $a\cdot b \leq |a||b|$. Hence, proving inequality (4.127) is equivalent to prove the numerical inequality

$$|a|^{2-p}|b|^p + |a|^p|b|^{2-p} + (1-c_p)(|a|-|b|)^2 - \left(\frac{|a|^{2-p}}{|b|^{2-p}} + \frac{|b|^{2-p}}{|a|^{2-p}}\right)|a||b| \geq 0,$$

when $|a| \geq |b|$. Dividing the above inequality by $|b|^2$ and letting $\lambda = |a|/|b|$, we get

$$\Phi_p(\lambda) = \lambda^{2-p} + \lambda^p + (1-c_p)(\lambda-1)^2 - \lambda^{3-p} - \lambda^{1-p} \geq 0 \qquad \text{for any } 1 < p \leq 2 \text{ and } \lambda \geq 1.$$

In the range $3/2 < p < 2$, we can always let $c_p = 1$, since $\lambda^{2-p} + \lambda^p \geq \lambda^{3-p} + \lambda^{1-p}$, and this guarantees that $\Phi_p(\lambda) \geq 0$; again this constant is optimal and achieved when $\lambda = 1$, that is when $a = b$. When $p = 3/2$, we have $\Phi_{3/2}(\lambda) = (1-c_p)(\lambda-1)^2 \geq 0$, so the inequality holds again with $c_p = 1$. When $1 < p < 3/2$ we have to work a bit more. We calculate

$$\Phi_p''(\lambda) = -(2-p)(p-1)\lambda^{-p} + p(p-1)\lambda^{p-2} - (3-p)(2-p)\lambda^{1-p} + (2-p)(p-1)\lambda^{p-3} + 2(1-c_p)$$

and we observe that $\Phi_p''(1) = -6 + 4p + 2(1-c_p) \geq 0$ if $c_p \leq 2(p-1)$. Moreover, in the limit $\lambda \to \infty$, $\Phi_p''(\lambda) \to 2(1-c_p) = 6 - 4p > 0$, when $1 < p < 3/2$. Then it is easy to check that

$$\begin{aligned}\Phi_p'''(\lambda) &= (p-1)(2-p)\left[p\lambda^{-p-1} - p\lambda^{p-3} + (3-p)\lambda^{-p} - (3-p)\lambda^{p-4}\right]\\ &\geq p(p-1)(p-2)\left(\frac{1}{\lambda}+1\right)\left(\frac{1}{\lambda^p} - \frac{1}{\lambda^{3-p}}\right) \geq 0\end{aligned}$$

since $3-p>p$ when $p<3/2$ and $t\geq 1$. We have thus proved that $\Phi_p''(\lambda)$ is a non-decreasing function of λ, which is zero in $\lambda=1$ and $\Phi_p(\lambda)\leq\Phi_p(\infty)=2(1-c_p)=6-4p$. This implies that $\lambda=1$ is a minimum for Φ_p, since $\Phi_p(1)=0$, $\Phi_p'(1)=0$. As a consequence $\Phi_p(\lambda)\geq 0$ for any $\lambda\geq 1$. Equality is attained for $\lambda=1$ and $c_p=2(p-1)$, and this fact proves optimality of the constant when $a=b$. $\square$

Chapter 5

Asymptotic behaviour of a nonlinear parabolic equation with gradient absorption and critical exponent

5.1 Introduction and main results

In this chapter we deal with the Cauchy problem associated to the diffusion-absorption equation:

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0\,, \quad (x,t) \in Q\,, \tag{5.1}$$

posed in $Q := \mathbb{R}^n \times (0,\infty)$ with initial data

$$u(x,0) = u_0(x) \geq 0\,, \quad x \in \mathbb{R}^n\,, \tag{5.2}$$

where the p-Laplacian operator is defined as usual by $\Delta_p u :=$ div $\left(|\nabla u|^{p-2}\, \nabla u\right)$. To be specific we take $p > 2$, which implies finite speed of propagation, and we consider nonnegative weak solutions $u \geq 0$ with compactly supported initial data u_0 such that

$$u_0 \in W^{1,\infty}(\mathbb{R}^n)\,, \;\; u_0 \geq 0\,, \quad \text{supp }(u_0) \subset B(0,R_0)\,, \;\; u_0 \not\equiv 0\,, \tag{5.3}$$

for some $R_0 > 0$. Known properties of the equation ensure that the corresponding solutions will be compactly supported with respect to the space variable for every time $t > 0$. The goal of the chapter is to describe in detail the asymptotic behaviour of the solutions as $t \to \infty$.

The equation (5.1) has been studied by various authors for different values of the parameters $p \geq 2$ and $q > 1$ as a model of linear or nonlinear diffusion with gradient-dependent absorption, see [20, 29, 64, 68, 100] for the semilinear case $p = 2$, and [4, 18, 99, 119] for the quasilinear case $p > 2$. It has been shown that the large-time behaviour of this initial-value problem depends on the relative influence of the diffusion and absorption terms and leads to a classification into the following ranges of q:

(i) when $q > q_2 := p - n/(n+1)$ the large time behaviour is purely diffusive and the first-order absorption term disappears in the limit $t \to \infty$; this is a case of asymptotic simplification in the sense of [122].
(ii) For $q_1 := p - 1 < q < q_2$ there is a behaviour given by a certain balance of diffusion and absorption in the form of a self-similar solution, its existence being established in [119]; there is no asymptotic simplification;
(iii) for $1 < q < p - 1$ the two last authors have recently shown in [99] that the main term is the absorption term, leading to a separate-variables asymptotic behaviour, with diffusion playing a secondary role. We thus have asymptotic simplification, now with absorption as the dominating effect.

The two critical cases $q = q_2$ and $q = q_1$ represent limit behaviours, and as is often the case in such situations, they give rise to interesting dynamics due to the curious interaction of two effects of similar power. Such situations usually lead to phenomena called *resonances* in mechanics, with interesting non-trivial mathematical analysis. Such interesting behaviour has been shown in particular in [64] for $q = q_2$, in the linear case $p = 2$, with the result that logarithmic factors modify the purely diffusive behaviour found for $q > q_2$. A similar situation is expected to be met when $p > 2$ and $q = q_2$.

We devote this chapter to describe the other limit case $q = q_1 = p - 1$ when $p > 2$, the latter condition guaranteeing that $q > 1$. In that case the diffusion and the first order term have similar asymptotic size and logarithmic corrections appear in the asymptotic rates. The mathematical analysis that we perform below is strongly tied to a good knowledge of the expansion of the support of the solution, or in other words, the location of the free boundary, which happens to be approximately a sphere of radius $|x| \sim C \log t$ for large times t. From now on, we assume that

$$q = q_1 = p - 1\,.$$

5.1.1 Bounds in suitable norms

Studying the large time behaviour of solutions and interfaces of our problem relies on suitable and very precise estimates. The time expansion of the support and the time decay of solutions to the Cauchy problem (5.1)-(5.2), with non-negative and compactly supported initial data have been recently investigated in [18]. The following results are proved:

Proposition 5.1. *Under the above assumptions on the equation and data, the Cauchy problem* (5.1) *has a unique non-negative viscosity solution*

$$u \in \mathcal{BC}(\mathbb{R}^n \times [0,\infty)) \cap L^\infty(0,\infty; W^{1,\infty}(\mathbb{R}^n))$$

which satisfies:

$$0 \le u(x,t) \le \|u_0\|_\infty\,, \quad (x,t) \in Q\,, \tag{5.4}$$

$$\|\nabla u(t)\|_\infty \le \|\nabla u_0\|_\infty\,, \quad t \ge 0\,, \tag{5.5}$$

$$\textit{supp}\ (u(t)) \subset B(0, C_1\ \log t) \quad \textit{for all} \quad t \ge 2\,, \tag{5.6}$$

together with the following norm estimates

$$\begin{aligned}
\|u(t)\|_1 &\leq C_2\, t^{-1/(p-2)}\, (\log t)^{(p(n+1)-2n-1)/(p-2)} \quad \textit{for all} \quad t \geq 2\,, &(5.7)\\
\|u(t)\|_\infty &\leq C_2\, t^{-1/(p-2)}\, (\log t)^{(p-1)/(p-2)} \quad \textit{for all} \quad t \geq 2\,, &(5.8)\\
\|\nabla u(t)\|_\infty &\leq C_2\, t^{-1/(p-2)}\, (\log t)^{1/(p-2)} \quad \textit{for all} \quad t \geq 2\,, &(5.9)
\end{aligned}$$

for some positive constants C_1 and C_2 depending only on p, n, and u_0.

Here and below, $\mathcal{BC}(\mathbb{R}^n \times [0,\infty))$ denotes the space of bounded and continuous functions on $\mathbb{R}^n \times [0,\infty)$ and $\|\cdot\|_r$ denotes the $L^r(\mathbb{R}^n)$-norm for $r \in [1,\infty]$. As we shall see, these bounds will be very useful in the sequel. The well-posedness of (5.1)-(5.2) and the properties (5.4), (5.6), and (5.7) are established in [18, Theorems 1.1 & 1.6, Corollary 1.7], while (5.8) and (5.9) follow from (5.7) and [18, Proposition 1.4]. We will also use the notation $r_+ = \max\{r, 0\}$ for the positive part of the real number r.

5.1.2 Main results

We describe next the main contribution of this chapter. As already mentioned, our goal is to study the asymptotic behaviour of the solution u of the resonant problem (5.1) with $p > 2$ and $q = p - 1$, and with compactly supported and nonnegative initial data. Moreover, since the equation has the property of finite speed of propagation, it is natural to raise the question about how the interface and the support of the solution expand in time. We also answer this question in the present study.

Asymptotic behaviour. The main result is the following:

Theorem 5.1. *Let u be the solution of the Cauchy problem* (5.1)-(5.2)*, with u_0 as in* (5.3)*. Then, u decays in time like $O(t^{-1/(p-2)}(\log t)^{(p-1)/(p-2)})$ and the support spreads in space like $O(\log t)$ as $t \to \infty$. More precisely, we have the limit:*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^n} \left| \frac{c_p\, t^{1/(p-2)}}{(\log t)^{(p-1)/(p-2)}}\; u\,(x,t) - \left(1 - \frac{(p-2)|x|}{\log\, t}\right)_+^{(p-1)/(p-2)} \right| = 0\,, \tag{5.10}$$

with precise constant

$$c_p = (p-2)^{1/(p-2)}(p-1)^{(p-1)/(p-2)}.$$

In the proof, the expression of the asymptotic profile is obtained after a complicated time scaling of u and x in the form of uniform limit

$$\frac{t^{1/(p-2)}}{(\log t)^{(p-1)/(p-2)}}\; u\,(x,t) \to (p-2)^{-p/(p-2)}\; W((p-2)x/\log t), \tag{5.11}$$

where the function

$$W(x) := \left(\frac{p-2}{p-1}\; (1-|x|)_+\right)^{(p-1)/(p-2)} \tag{5.12}$$

is the unique viscosity solution to the stationary form of the rescaled problem, which is:

$$|\nabla W|^{p-1} - W = 0 \ \text{ in } \ B(0,1)\,, \quad W = 0 \ \text{ on } \ \partial B(0,1), \quad W > 0 \ \text{ in } \ B(0,1). \tag{5.13}$$

Let us notice that, as usual in resonance cases, the limit profile is not a self-similar solution, but it introduces logarithmic corrections to a self-similar, separate-variables profile (which in our case is $t^{-1/(p-2)}(p-2)^{-p/(p-2)}W((p-2)x)$). The uniqueness of W as viscosity solution of (5.13) is very important in the proof and follows from [85].

In consonance with (5.10), we show that the shape of the support of $u(t)$ gets closer to a ball while expanding as time goes by. This is in sharp contrast with the situation described in [99] for (5.1) in the intermediate range $q \in (1, p-1)$, $p > 2$ where the positivity set stays bounded and can have a very general shape. When $q = p-1$, the diffusion thus acts in three directions: the scaling is different, the support grows unboundedly with time, and the geometry of the positivity set simplifies. Another remarkable consequence of the interplay diffusion-absorption is the fact that the asymptotic profile is radially symmetric and does not depend on the space dimension.

We devote Section 5.4 to the proof of Theorem 5.1. For the proof, we use a precise estimate for the propagation of the positivity set, that is described below. Another tool is the existence of a large family of subsolutions having a special, explicit form and allowing for a theoretical argument with viscosity solutions to finish the proof.

Propagation of the positivity set. We denote the positivity set and its maximal expansion radius by

$$\mathcal{P}_u(t) := \{x \in \mathbb{R}^n : \ u(x,t) > 0\}, \qquad \gamma(t) = \sup\{|x| : \ x \in P_u(t)\} \tag{5.14}$$

respectively. Then:

Theorem 5.2. *Under the running notations and assumptions, we have:*

$$\lim_{t\to\infty} \frac{\gamma(t)}{\log t} = \frac{1}{p-2}. \tag{5.15}$$

Moreover, the free boundary of u has the same speed of expansion in any given direction $\omega \in \mathbb{R}^n$ with $|\omega| = 1$.

In fact, we give more precise estimates for the expansion of the positivity region, obtained via comparison with some well-chosen traveling waves. The proof of Theorem 5.2 is performed in Section 5.3.

Two scalings. In order to prove the two theorems, we have to perform two different scaling steps. The first scaling, described in formula (5.17) below, is the natural one corresponding to standard scaling invariance; such a scaling has also been used in [99] in the case $q \in (1, p-1)$ to obtain the correct scale of the solutions. But for $q = p-1$, we observe that a phenomenon of grow-up appears, which is typical for resonance cases: the effect of the resonance implies that the rescaled solution does not stabilize in time; on the contrary, it grows and becomes unbounded in infinite time. That is why we need a second scaling, given by the new functions w and y defined in (5.45) and (5.46), which is less natural but turns out to be adapted to our problem: it takes into account the logarithmic corrections (suggested by the a priori estimates of Proposition 5.1, which turn out to be sharp), and it is adapted to the size of the grow-up phenomenon; thus, in the rescaled variables we can describe the real form and behaviour of the solution.

5.2 Scaling variables I

We recall that $p > 2$ and $q = p - 1$. We introduce a first set of *self-similar* variables; we keep the space variable x and introduce logarithmic time

$$\tau := \frac{1}{p-2} \ln\left(1 + (p-2)t\right), \tag{5.16}$$

as well as the new unknown function $v = v(x, \tau)$ defined by

$$u(x,t) = (1 + (p-2)t)^{-1/(p-2)}\ v\left(x, \tau\right)\ , \quad (x,t) \in \mathbb{R}^n \times [0, \infty)\,. \tag{5.17}$$

Clearly, v solves the rescaled equation

$$\partial_\tau v - \Delta_p v + |\nabla v|^q - v = 0\ \ , \quad (x,\tau) \in Q\,, \tag{5.18}$$

with the same initial condition

$$v(0) = u_0\,, \quad x \in \mathbb{R}^n\,. \tag{5.19}$$

We next translate the a priori bounds (5.7), (5.8), and (5.9) in terms of the rescaled function v: there is $C_3 > 0$ depending only on p, n, and u_0 such that

$$\frac{\|v(\tau)\|_1}{\tau^{(p(n+1)-2n-1)/(p-2)}} + \frac{\|v(\tau)\|_\infty}{\tau^{(p-1)/(p-2)}} + \frac{\|\nabla v(\tau)\|_\infty}{\tau^{1/(p-2)}} \le C_3 \quad \text{for} \quad \tau \ge 1\,. \tag{5.20}$$

5.2.1 The positivity set: time monotonicity

We define the positivity set $\mathcal{P}_v(\tau)$ of the function v at time $\tau \ge 0$ by

$$\mathcal{P}_v(\tau) := \{x \in \mathbb{R}^n :\ \ v(x,\tau) > 0\}\ . \tag{5.21}$$

Proposition 5.2. *For $\tau_1 \in [0, \infty)$ and $\tau_2 \in (\tau_1, \infty)$ we have*

$$\mathcal{P}_v(\tau_1) \subseteq \mathcal{P}_v(\tau_2) \quad \text{ and } \quad \bigcup_{\tau \ge 0} \mathcal{P}_v(\tau) = \mathbb{R}^n\,. \tag{5.22}$$

In addition, for each $x \in \mathbb{R}^n$ there are $T_x \ge 0$ and $\varepsilon_x > 0$ such that

$$v(x,\tau) \ge \varepsilon_x\ \tau^{(p-1)/(p-2)} \quad \text{for} \quad \tau \ge T_x\,. \tag{5.23}$$

The proof relies on the availability of suitable subsolutions which we describe next.

Lemma 5.1. *Define two positive real numbers R_p and T_p by*

$$R_p := \frac{p-2}{2^p(p-1)} \quad \text{and} \quad T_p := \frac{2(p-1)}{p-2} \ \left(2+2^{p-1}(n+p-2)\right) .$$

If $R \in (0,R_p]$ and $T \geq T_p$, the function $s_{R,T}$ given by

$$s_{R,T}(x,\tau) := \frac{p-2}{R(p-1)} \ (T+\tau)^{(p-1)/(p-2)} \ \left(R^2 - \frac{|x|^2}{(T+\tau)^2}\right)_+^{(p-1)/(p-2)} , \quad (x,\tau)\in Q ,$$

is a (viscosity) subsolution to (5.18).

Proof. We have $s_{R,T}(x,\tau) = (T+\tau)^{(p-1)/(p-2)} \ \sigma(\xi)$ with $\xi := x/(T+\tau)$ and $\sigma(\xi) := (p-2) \ \left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)}/(R(p-1))$. Since $p-1>p-2>0$, we observe that σ and $|\nabla\sigma|^{p-2}\nabla\sigma$ both belong to $\mathcal{C}^1(\mathbb{R}^n)$. Therefore,

$$L(x,\tau) := R \ (T+\tau)^{-(p-1)/(p-2)} \ \left\{\partial_\tau s_{R,T} - \Delta_p s_{R,T} + |\nabla s_{R,T}|^{p-1} - s_{R,T}\right\}$$

is well-defined for $(x,\tau)\in\mathbb{R}^n\times[0,\infty)$ and

$$\begin{aligned}
L(x,\tau) &= \frac{R}{T+\tau} \ \left\{\frac{p-1}{p-2} \ \sigma(\xi) - \xi\cdot\nabla\sigma(\xi) - \Delta_p\sigma(\xi)\right\} + R \ |\nabla\sigma(\xi)|^{p-1} - R \ \sigma(\xi) \\
&= \left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)} \ \left\{\frac{1}{T+\tau} \ \left(1+2^{p-1}(n+p-2) \ \frac{|\xi|^{p-2}}{R^{p-2}}\right)\right\} \\
&+ \left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)} \ \left\{\frac{2}{T+\tau} \ \frac{|\xi|^2}{R^2-|\xi|^2} \ \left(1-\frac{2^{p-1}(p-1)}{p-2} \ \frac{|\xi|^{p-2}}{R^{p-2}}\right)\right\} \\
&+ \left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)} \ \left\{2^{p-1}\frac{|\xi|^{p-1}}{R^{p-2}} - \frac{p-2}{p-1}\right\} \\
&\leq \left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)} \ \left\{\frac{1+2^{p-1}(n+p-2)}{T} + 2^{p-1}R - \frac{p-2}{p-1}\right\} \\
&+ \left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)} \ \left\{\frac{2}{T+\tau} \ \frac{|\xi|^2}{R^2-|\xi|^2} \ \left(1-\frac{2^{p-1}(p-1)}{p-2} \ \frac{|\xi|^{p-2}}{R^{p-2}}\right)_+\right\} .
\end{aligned}$$

We next note that

$$1-\frac{2^{p-1}(p-1)}{p-2} \ \frac{|\xi|^{p-2}}{R^{p-2}} \leq 0 \quad \text{if} \quad |\xi| \geq \frac{R}{2} ,$$

so that the last term of the right-hand side of the previous inequality is bounded from above by $2\left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)}/(3T)$. Consequently, owing to the choice of R and T,

$$\begin{aligned}
L(x,\tau) &\leq \left(R^2-|\xi|^2\right)_+^{(p-1)/(p-2)} \ \left\{\frac{1+2^{p-1}(n+p-2)}{T_p} + 2^{p-1}R_p - \frac{p-2}{p-1} + \frac{2}{3T_p}\right\} \\
&\leq 0 ,
\end{aligned}$$

whence the claim. □

Proof of Proposition 5.2. (i) Fix $\tau_1 \geq 0$ and $x_1 \in \mathcal{P}_v(\tau_1)$. Owing to the continuity of $x \longmapsto v(x,\tau_1)$ there are $\delta > 0$ and $r_1 > 0$ such that $v(x,\tau_1) \geq \delta$ for $x \in B(x_1, r_1)$. Take now $R > 0$ small enough such that $R < \min\{r_1, R_p\}$ and satisfying

$$R < \frac{r_1}{T_p + \tau_1} \qquad \text{and} \qquad \frac{p-2}{p-1}\,(T_p+\tau_1)^{(p-1)/(p-2)}\ R^{p/(p-2)} \leq \delta\,,$$

the parameters R_p and T_p being defined in Lemma 5.1. Then we have $s_{R,T_p}(x - x_1, \tau_1) = 0 \leq v(x,\tau_1)$ if $|x - x_1| \geq R\ (T_p + \tau_1)$, while

$$s_{R,T_p}(x-x_1,\tau_1) \leq \frac{p-2}{R(p-1)}\ (T_p+\tau_1)^{(p-1)/(p-2)}\ R^{(2p-2)/(p-2)} \leq \delta \leq v(x,\tau_1)$$

if $|x - x_1| \leq R\ (T_p + \tau_1)$ as $R(T_p + \tau_1) \leq r_1$. Moreover, if $\tau_2 > \tau_1$, $\tau \in [\tau_1, \tau_2]$ and $x \in \partial B(x_1, R(T_p+\tau_2))$, then $s_{R,T_p}(x - x_1, \tau) = 0 \leq v(x,\tau)$. Recalling that s_{R,T_p} is a subsolution to (5.18) by Lemma 5.1, we infer from the comparison principle that $s_{R,T_p}(x-x_1,\tau) \leq v(x,\tau)$ for $(x,\tau) \in B(x_1, R(T_p+\tau_2)) \times [\tau_1,\tau_2]$. As $s_{R,T_p}(x - x_1, \tau) = 0 \leq v(x,\tau)$ for $\tau \in [\tau_1, \tau_2]$ and $x \notin B(x_1, R(T_p + \tau_2))$ we actually have $s_{R,T_p}(x - x_1, \tau) \leq v(x,\tau)$ for $(x,\tau) \in \mathbb{R}^n \times [\tau_1,\tau_2]$. Since $\tau_2 > \tau_1$ is arbitrary and neither R nor T_p depend on τ_2, we end up with

$$s_{R,T_p}(x-x_1,\tau) \leq v(x,\tau)\,, \qquad (x,\tau) \in \mathbb{R}^n \times [\tau_1,\infty)\,. \tag{5.24}$$

A first consequence of (5.24) is that, if $\tau_2 > \tau_1$, then $v(x_1,\tau_2) \geq s_{R,T_p}(0,\tau_2) > 0$, so that x_1 also belongs to $\mathcal{P}_v(\tau_2)$.

Next, given $x \in \mathbb{R}^n$, we have $x \in B(x_1, R(T_p + \tau))$ for τ large enough and it follows from (5.24) that $v(x,\tau) \geq s_{R,T_p}(x - x_1, \tau) > 0$ for τ large enough. Consequently, x belongs to $\mathcal{P}_v(\tau)$ for τ large enough which proves the second assertion of (5.22).

(ii) Consider $x_0 \in \mathbb{R}^n$. According to (5.22) there is τ_0 large enough such that $x_0 \in \mathcal{P}_v(\tau_0)$. Arguing as in the proof of (5.22), we may find r_0 small enough (depending on x_0) such that $s_{r_0,T_p}(x - x_0, \tau) \leq v(x,\tau)$ for $(x,\tau) \in \mathbb{R}^n \times [\tau_0,\infty)$. Consequently,

$$v(x_0,\tau) \geq \frac{p-2}{r_0(p-1)}\ (T_p+\tau)^{(p-1)/(p-2)}\ r_0^{(2p-2)/(p-2)} \geq \frac{p-2}{p-1}\ r_0^{p/(p-2)}\ \tau^{(p-1)/(p-2)}\,,$$

which gives the lower bound (5.23). □

Corollary 5.1. *Assume that $u_0(0) > 0$. Then there is $r_* > 0$ such that*

$$v(x,\tau) \geq \frac{(p-2)}{r_*(p-1)}(1+\tau)^{(p-1)/(p-2)}\left(r_*^2 - \frac{|x|^2}{(1+\tau)^2}\right)_+^{(p-1)/(p-2)}\,, \quad (x,\tau) \in Q\,. \tag{5.25}$$

Proof. Arguing as in the proof of (5.22) and using the positivity of $u_0(0)$, we may find $r_* > 0$ small enough such that $s_{r_*,T_p}(x,\tau) \leq v(x,\tau)$ for $(x,\tau) \in Q$. Since $T_p > 1$, the previous inequality implies (5.25). □

5.2.2 Eventual radial symmetry

We prove the following classical monotonicity lemma, see [6, Proposition 2.1] for instance.

Lemma 5.2. *If $x \in \mathbb{R}^n$ and $r > 0$ satisfy $|x| > 2R_0$ and $r < |x| - 2R_0$, then,*

$$v(x,\tau) \le \inf_{|y|=r} v(y,\tau) \quad \textit{for} \quad \tau \ge 0\,. \tag{5.26}$$

Here, R_0 is radius of the initial ball defined in (5.3).

Proof. The proof relies on Alexandrov's reflection principle. Let $(x,r) \in \mathbb{R}^n \times (0,\infty)$ fulfil the assumptions of Lemma 5.2 and consider $y \in \mathbb{R}^n$ such that $|y| = r$. Let H be the hyperplane of points of $\mathbb{R}^n$ which are equidistant from x and y, namely

$$H := \left\{ z \in \mathbb{R}^n \ : \ \left\langle z - \frac{x+y}{2}, x-y \right\rangle = 0 \right\}\,.$$

Introducing

$$H_- := \left\{ z \in \mathbb{R}^n \ : \ \left\langle z - \frac{x+y}{2}, x-y \right\rangle \le 0 \right\}$$

and

$$\tilde{v}(z,\tau) := v\left(z - 2\,\left\langle z - \frac{x+y}{2}, x-y \right\rangle\, \frac{x-y}{|x-y|^2}, \tau \right)\,, \qquad (z,\tau) \in Q\,,$$

it readily follows from the rotational and translational invariance of (5.18) that $\tilde{v}$ also solves (5.18). In addition, $y \in H_-$ and $\mathcal{P}_v(0) \subseteq B(0,R_0) \subseteq H_-$ by (5.3). Now, on the one hand, if $z \in H_-$, then

$$z - 2\,\left\langle z - \frac{x+y}{2}, x-y \right\rangle\, \frac{x-y}{|x-y|^2} \notin H_-$$

and $\tilde{v}(z,0) = 0 \le v(z,0)$. On the other hand, if $z \in H = \partial H_-$ and $\tau \ge 0$, we clearly have $\tilde{v}(z,\tau) = v(z,\tau)$. We are then in a position to apply the comparison principle to (5.18) on $H_- \times (0,\infty)$ and conclude that

$$\tilde{v}(z,\tau) \le v(z,\tau)\,, \qquad (z,\tau) \in H_- \times [0,\infty)\,. \tag{5.27}$$

Recalling that $y \in H_-$, we infer from (5.27) that $v(y,\tau) \ge \tilde{v}(y,\tau) = v(x,\tau)$ for $\tau \ge 0$ which is the expected result. □

Remark 1. *Although Lemma 5.2 will not be used in the main proofs, this is an interesting result for the qualitative theory, since it shows that the dynamics symmetrizes the solution.*

5.3 Propagation of the positivity set

We next turn to the speed of expansion of the positivity set $\mathcal{P}_v$ of v and put

$$\varrho(\tau) := \sup\left\{ |x| \ : \ x \in \mathcal{P}_v(\tau) \right\}, \tag{5.28}$$

so that $\mathcal{P}_v(\tau) \subseteq B(0, \varrho(\tau))$ for $\tau \geq 0$. The purpose of this section is to prove that the expansion speed $\varrho(\tau)$ of $\mathcal{P}_v(\tau)$ is asymptotically equal to τ, in other words,

$$\lim_{\tau\to\infty} \frac{\varrho(\tau)}{\tau} = 1,$$

and, more precisely, to prove Theorem 5.2.

The proof relies on the existence of "nice" traveling wave solutions of (5.18), which may be used as subsolutions and supersolutions for the Cauchy problem (5.18)-(5.19). The construction of such traveling waves is inspired on the technique used in the so-called KPP problems, [96], which has developed a wide literature; see e. g., [5], [123] for applications to porous media, and [113] for blow-up problems. We thus begin with a phase-plane analysis, proving the existence of the desired traveling waves.

5.3.1 Traveling wave analysis for $n = 1$

We look for traveling waves of the form

$$v(x,\tau) = f(z), \quad z = x - c\tau, \ c > 0,$$

solving (5.18) in dimension $n = 1$. Then, the profile f solves the ordinary differential equation:

$$-cf' - \left(|f'|^{p-2}f'\right)' + |f'|^{p-1} - f = 0. \tag{5.29}$$

We are actually only interested in traveling waves which present an interface, that is, f vanishes for z sufficiently large. As we shall see below, the profile f is non-monotone in general, but is nonnegative and decreasing near the interface. We transform (5.29) into a first order system, by introducing the notation $U = f$ and $V = -f'$. We arrive at the following system

$$\begin{cases} (p-1)|V|^{p-2}U' = -(p-1)|V|^{p-2}V, \\ (p-1)|V|^{p-2}V' = -cV - |V|^{p-1} + U, \end{cases} \tag{5.30}$$

where, for the orbits, the term $(p-1)|V|^{p-2}$ in the right-hand side has no influence (since we work with dV/dU) and can be ignored after a change of the time variable. We perform next the phase-plane analysis of the system (5.30).

Local analysis in the plane. The system (5.30) has a unique critical point, $P = (0,0)$, and the Jacobian matrix $J(0,0)$ at this point is given by

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & -c \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -c$, and corresponding eigenvectors $e_1 = (c,1)$ and $e_2 = (0,1)$. By a careful analysis, we notice that the center manifold in P is tangent to e_1, and is asymptotically stable. It follows that P is a stable node for every $c > 0$. There is a unique orbit entering P and tangent to e_2, forming the stable manifold; its local behaviour is $U(z) \sim C(-z)^{(p-1)/(p-2)}$ as $z \to 0$, hence this orbit contains all the traveling waves with velocity c and having an interface. By standard theory (see, e.g., [109]), all the other orbits approach the center manifold, tangent to e_1, and present an exponential decay, but no interface: $U(z) \sim e^{-cz}$ as $z \to \infty$.

Local analysis at infinity. We investigate the behaviour of the system when U is very large. For monotone traveling waves, we make the following inversion of the plane:

$$Z = \frac{1}{U}, \quad W = \frac{|V|^{p-2}V}{U},$$

and we are interested in the local behaviour near $Z = 0$. After straightforward calculations, (5.30) becomes the new system:

$$\begin{cases} Z' = Z^{(2p-3)/(p-1)}W|W|^{-(p-2)/(p-1)}, \\ W' = Z^{(p-2)/(p-1)}|W|^{p/(p-1)} - cZ^{(p-2)/(p-1)}W|W|^{-(p-2)/(p-1)} + 1 - |W|. \end{cases} \tag{5.31}$$

We find two critical points with $Z = 0$, namely $Q_1 = (0, 1)$ and $Q_2 = (0, -1)$. We will analyze only Q_1, i.e. the decreasing traveling waves. Let us also remark that, in the second equation of (5.31), the terms with Z are dominated by $1 - |W|$ near Q_1 and Q_2, hence we can study the local behaviour by using the approximate equation only with $1 - |W|$ in the right-hand side. The linearization near Q_1 has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -1$, and the center manifold, which is tangent to the line $W = 1$, is unstable. Hence, the point Q_1 behaves like a saddle, and the orbits which are interesting for our study are the orbits going out of Q_1. These orbits are tangent to $W = 1$, and in the original system they satisfy $U \sim V^{p-1}$, hence, by integration,

$$U(z) \sim |z|^{(p-1)/(p-2)}, \quad \text{as } z \to -\infty,$$

and are decreasing. The local analysis around Q_2 is similar, but not interesting for our goals.

Let us notice that not all solutions passing through a point in the first quadrant come from Q_1. Indeed, the orbits touching the curve $U = cV + V^{p-1}$ change monotonicity as functions $V = V(U)$, hence they have previously reached the axis $V = 0$, meaning a change of monotonicity as $f = f(z)$, and they enter through this change in the first quadrant. Analyzing the curve $U = cV + V^{p-1}$, we observe that it connects in the phase-plane the points $P = (0, 0)$ and Q_1, being tangent in Q_1 to the axis $W = 1$. In particular, there exist non-monotone solutions, and this is the object we are interested in.

Global behaviour. This is now not difficult to establish, by merging the previous local analysis with the following important remarks:

(a) The evolution of the system (5.30) with respect to the parameter c is monotone. Indeed, we calculate:

$$\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\mathrm{d}V}{\mathrm{d}U}\right) = \frac{1}{(p-1)|V|^{p-2}} > 0.$$

(b) There exists an explicit family of traveling wave solutions with speed $c = 1$:

$$f_{1,K}(z) = \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (K - z)_+^{(p-1)/(p-2)}, \quad K \geq 0. \tag{5.32}$$

This function is obviously decreasing and presents an interface at $z = K$. It is immediate to check that this orbit satisfies $U = V^{p-1}$, hence it comes from the point Q_1 along the center manifold of it, and it enters P, being the unique orbit entering P and tangent to the eigenvector $e_2 = (0, 1)$ (unique for $c = 1$), as discussed above.

(c) Moreover, the vectors of the direction field of (5.30) over the curve $U = V^{p-1}$ (which gives the explicit orbit (5.32)) have the same direction. Indeed, the normal vector to this curve is $(1, -(p-1)V^{p-2})$ and we calculate:

$$(1, -(p-1)V^{p-2}) \cdot (-(p-1)V^{p-1}, -cV - V^{p-1} + U) = (p-1)(c-1)V^{p-1}.$$

For $c = 1$ we obtain the explicit trajectory, and for $c < 1$, the above scalar product is negative, hence all these vectors have the same direction, contrary to $(1, -(p-1)V^{p-2})$. For $c > 1$, all these vectors have the same direction as V.

Since we are interested only in traveling waves with an interface, we analyze only the unique (for c fixed) orbit entering $P = (0, 0)$ tangent to $e_2 = (0, 1)$. For $c = 1$, it is explicit and connects P and Q_1 in the first quadrant. We draw the phase-plane for $c = 1$ in Figure 5.1 below; it is clear that the explicit connection will not change sign and monotonicity.

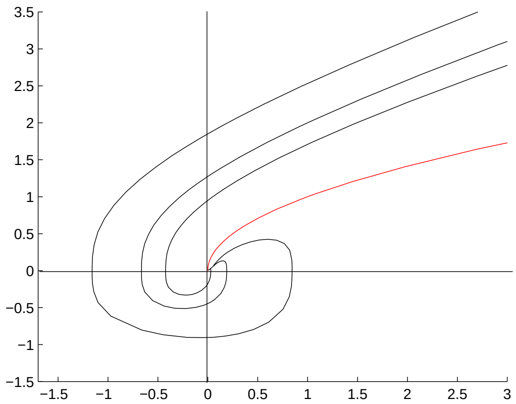

Figure 5.1: Phase portrait around the origin for $c = 1$. Experiment for $p = 3$, $n = 2$.

By remarks (a) and (c) above, it follows that for $c < 1$, this unique orbit disconnects from Q_1, hence it should cross at some point the curve $U = cV + V^{p-1}$ (which still connects $P = (0, 0)$ and Q_1); as explained before, this orbit previously had a change of sign (crossing the axis $U = 0$) and then a change of monotonicity (crossing the axis $V = 0$). In particular, we can say that the explicit orbit (5.32) is a separatrix between the monotone and the non-monotone orbits. We draw the local phase portrait for $c < 1$, around the origin, in Figure 5.2 below. We gather the discussion above in the following result.

Lemma 5.3. *(i) For any $c \in (0, 1)$ and $K \geq 0$, there exists a unique traveling wave solution $\overline{f}_{c,K}(z) = \overline{f}_{c,K}(x - c\tau)$ of (5.18) in dimension $n = 1$, having an interface at $z = K$ (that*

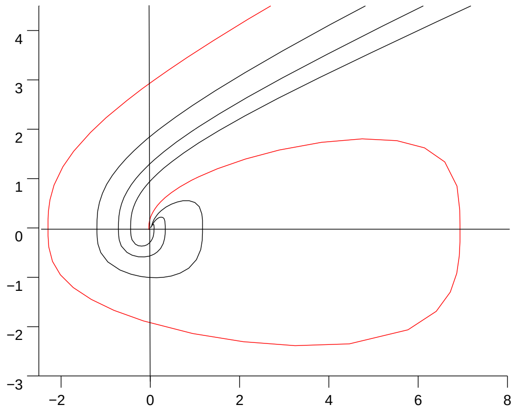

Figure 5.2: Phase portrait around the origin for $c<1$. Experiment for $p=3$, $n=2$, $c=0.9$.

is, $\overline{f}_{c,K}(z)=0$ *for* $z\geq K$*) and moving with speed* c*. In addition,* $\overline{f}_{c,K}(z)=\overline{f}_{c,0}(z-K)$ *for* $z\in\mathbb{R}$*.*
(ii) For $c=1$ *and for any* $K\geq 0$*, there exists a unique nonnegative traveling wave* $f_{1,K}(z)=f_{1,K}(x-\tau)$ *of* (5.18) *in dimension* $n=1$ *with interface at* $z=K$*, having the explicit formula:*

$$f_{1,K}(x-\tau)=\left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)}(K+\tau-x)_+^{(p-1)/(p-2)}. \tag{5.33}$$

Here again, $f_{1,K}(z)=f_{1,0}(z-K)$ *for* $z\in\mathbb{R}$*.*
(iii) For any $c>1$ *and* $K\geq 0$*, there exists a unique traveling wave solution* $f_{c,K}=f_{c,K}(x-c\tau)$ *of* (5.18) *in dimension* $n=1$ *with interface at* $z=K$ *and moving with speed* c*. Moreover,* $f_{c,K}$ *is nonnegative and non-increasing, and* $f_{c,K}(z)=f_{c,0}(z-K)$ *for* $z\in\mathbb{R}$*.*

Compactly supported subsolutions for $0<c<1$**.** We are looking for nonnegative and compactly supported subsolutions traveling with any speed $0<c<1$. These subsolutions are constructed in the following way: from the analysis above, we know that, given $c\in(0,1)$ and $K\geq 0$, there are two points $z_{c,K}\in(-\infty,K)$ and $\tilde{z}_{c,K}\in(z_{c,K},K)$ such that

$$z_{c,K}:=\inf\left\{z\in(-\infty,K)\ :\ \overline{f}_{c,K}>0\ \text{ in }\ (z,K)\right\}>-\infty\,,$$

and

$$\overline{f}'_{c,K}>0\ \text{ in }\ (z_{c,K},\tilde{z}_{c,K})\quad\text{ and }\quad\overline{f}'_{c,K}<0\ \text{ in }\ (\tilde{z}_{c,K},K)\,.$$

We then define

$$f_{c,K}(z)=\left\{\begin{array}{ll}\overline{f}_{c,K}(z), & \text{for } z_{c,K}\leq z\leq K,\\ 0, & \text{elsewhere.}\end{array}\right. \tag{5.34}$$

In other words, we consider the positive hump of the graph of $f_{c,K}$ located between its last change of sign and the interface. It is immediate to check that $f_{c,K}$ is a compactly supported subsolution to (5.18) in dimension $n = 1$, and that it has an increasing part until reaching its maximum at $\tilde{z}_{c,K}$, and then decreases to the interface point K. The notation $f_{c,K}$ will designate in the sequel these subsolutions if $0 < c < 1$ and the solutions to (5.18) in dimension $n = 1$ given by Lemma 5.3 if $c \geq 1$.

5.3.2 Construction of subsolutions in dimension $n \geq 1$

We turn to equation (5.18) posed in dimension $n \geq 1$ for which we aim at constructing some special subsolutions having an interface that moves out in all directions with a given velocity $c < 1$. The construction is based on the traveling waves $f_{c,K}$ identified in the previous subsection. The first attempt is to try the form $V(x,\tau) = f_{c,K}(|x| - c\tau)$, $c \in (0,1)$, which satisfies:

$$\begin{aligned}
& \partial_\tau V - \Delta_p V + |\nabla V|^{p-1} - V \\
= \quad & -c f'_{c,K} - \left(|f'_{c,K}|^{p-2} f'_{c,K}\right)' + |f'_{c,K}|^{p-1} - f_{c,K} - \frac{n-1}{|x|} |f'_{c,K}|^{p-2} f'_{c,K} \\
\leq \quad & -\frac{n-1}{|x|} |f'_{c,K}|^{p-2} f'_{c,K} \,.
\end{aligned}$$

Thus, V is a subsolution of (5.18) in the region of Q where $f'_{c,K} \geq 0$. We therefore have to modify the profile in the decreasing part of $f_{c,K}$ and we proceed as follows.

Traveling wave solutions to a modified equation in dimension $n = 1$. For $\alpha \in (0, 1/2)$, we consider the following perturbation of (5.18):

$$\partial_\tau \zeta - \partial_x \left(|\partial_x \zeta|^{p-2} \partial_x \zeta\right) + |\partial_x \zeta|^{p-1} - \alpha\, |\partial_x \zeta|^{p-2} \partial_x \zeta - \zeta = 0\,, \quad (x,t) \in \mathbb{R} \times (0,\infty)\,, \tag{5.35}$$

and look for traveling wave solutions $\zeta(x,\tau) = g(x - c\tau)$. Then, g solves

$$-cg' - \left(|g'|^{p-2} g'\right)' + |g'|^{p-1} - \alpha\, |g'|^{p-2} g' - g = 0. \tag{5.36}$$

The phase-plane analysis for (5.36) is similar to that of (5.29), with the difference that an extra term $-\alpha\, |V|^{p-2} V$ appears in the right-hand side of the second equation in (5.30). This is only reflected in the analysis at infinity, where the point Q_1 changes into $(0, 1/(1+\alpha))$ and the explicit separatrix is obtained for $c = 1/(1+\alpha) < 1$. In particular, we have the following analogue of Lemma 5.3 (i).

Lemma 5.4. *For any $\alpha > 0$ sufficiently small, $c \in (0, 1/(1+\alpha))$ and $K \geq 0$, there exists a unique traveling wave solution $g_{c,K,\alpha}(z) = g_{c,K,\alpha}(x - c\tau)$ of (5.35) having an interface at $z = K$ and moving with speed c. In addition, $g_{c,K,\alpha}(z) = g_{c,0,\alpha}(z - K)$ for $z \in \mathbb{R}$ and there are two points $z_{c,K,\alpha} \in (-\infty, K)$ and $\tilde{z}_{c,K,\alpha} \in (z_{c,K,\alpha}, K)$ such that*

$$z_{c,K,\alpha} := \inf \{z \in (-\infty, K) \ : \ g_{c,K,\alpha} > 0 \ \text{ in } \ (z,K)\} > -\infty\,,$$

and

$$g'_{c,K,\alpha} > 0 \ \text{ in } \ (z_{c,K,\alpha}, \tilde{z}_{c,K,\alpha}) \quad \text{and} \quad g'_{c,K,\alpha} < 0 \ \text{ in } \ (\tilde{z}_{c,K,\alpha}, K)\,.$$

Setting

$$M_{c,\alpha} := \sup_{z\in[z_{c,0,\alpha},0]} \{g_{c,0,\alpha}(z)\}\,,$$

we notice that

$$z_{c,K,\alpha} = z_{c,0,\alpha} + K\,, \quad \tilde{z}_{c,K,\alpha} = \tilde{z}_{c,0,\alpha} + K\,, \quad \sup_{z\in[z_{c,K,\alpha},K]} \{g_{c,K,\alpha}(z)\} = M_{c,\alpha}\,. \tag{5.37}$$

If we put now $V(x,\tau) = g_{c,K,\alpha}(|x| - c\tau)$, we calculate and find that

$$\partial_\tau V - \Delta_p V + |\nabla V|^{p-1} - V = \left(\alpha - \frac{n-1}{|x|}\right) \left(|g'_{c,K,\alpha}|^{p-2} g'_{c,K,\alpha}\right)(|x| - c\tau)\,,$$

and it is a subsolution where $g'_{c,K,\alpha} \le 0$ and $\alpha \ge (n-1)/|x|$. Matching these two conditions turns out to be possible as we show now.

Fix $c \in (1/2,1)$ and $\alpha_c := (1-c)/(1+c)$ and define

$$\tau_0(c) := \max\left\{\frac{2(n-1)}{\alpha_c} - 2\tilde{z}_{c,0,\alpha_c}, -\frac{\tilde{z}_{c,0,\alpha_c}}{c}\right\} > \frac{2(n-1)}{\alpha_c}\,, \tag{5.38}$$

the point $\tilde{z}_{c,0,\alpha_c} \in (-\infty,0)$ being defined in Lemma 5.4. Then $c < 1/(1+\alpha_c)$ and, for $K \ge 0$, $\tau \ge \tau_0(c)$, and $|x| \ge \tilde{z}_{c,K,\alpha_c} + c\tau = \tilde{z}_{c,0,\alpha_c} + K + c\tau$, we have

$$\frac{n-1}{|x|} \le \frac{n-1}{\tilde{z}_{c,0,\alpha_c} + c\tau_0(c)} \le \frac{2(n-1)}{2\tilde{z}_{c,0,\alpha_c} + \tau_0(c)} \le \alpha_c\,,$$

and

$$\begin{aligned} &g'_{c,K,\alpha_c}(|x| - c\tau) < 0 \quad \text{if} \quad \tilde{z}_{c,K,\alpha_c} + c\tau \le |x| < K + c\tau\,,\\ &g'_{c,K,\alpha_c}(|x| - c\tau) = 0 \quad \text{if} \quad K + c\tau \le |x|\,. \end{aligned}$$

Consequently, for $c \in (1/2,1)$, $\alpha_c = (1-c)/(1+c)$, and $K > 0$, the function V defined by $V(x,\tau) = g_{c,K,\alpha_c}(|x| - c\tau)$ is a subsolution to (5.18) for $\tau \ge \tau_0(c)$ and $|x| \ge \tilde{z}_{c,K,\alpha_c} + c\tau$. Observing that any positive constant is a subsolution to (5.18), we construct a compactly supported subsolution $v_{c,K}$ to (5.18) by setting

$$v_{c,K}(x,\tau) := \begin{cases} M_{c,\alpha_c} & \text{if } 0 \le |x| < \tilde{z}_{c,K,\alpha_c} + c\tau\,,\\ \\ g_{c,K,\alpha_c}(|x| - c\tau) & \text{if } |x| > \tilde{z}_{c,K,\alpha_c} + c\tau\,, \end{cases} \tag{5.39}$$

for $\tau \ge \tau_0(c)$. It is easy to check that the function $v_{c,K}$ is a subsolution to (5.18) in $\mathbb{R}^n \times [\tau_0(c),\infty)$. It will be used for comparison from below, as indicated in the next subsection.

5.3.3 Proof of Theorem 5.2

We conclude the proof of Theorem 5.2 by a comparison argument, using the subsolutions and supersolutions constructed in the previous subsections. Before that, we identify a class of solutions of (5.18) that is representative for the general solutions.

We say that a function $V = V(x,\tau)$ is *radially non-increasing* if $V(\cdot,\tau)$ is radially symmetric for all τ, and it is non-increasing in the radial variable $r := |x|$. For example, the subsolutions $v_{c,K}$ are radially non-increasing. The next results show that the class of radially non-increasing solutions of (5.18) is sufficient for our aims.

Lemma 5.5. *Let $u_0 = u_0(r)$ be a radially non-increasing function satisfying* (5.3)*. Then, the solution v of* (5.18) *with initial condition u_0 is also radially non-increasing.*

Proof. The radial symmetry of the solution v follows from the invariance of the equation (5.18) with respect to rotations. We write now the equation satisfied by $\xi = \partial_r v$, obtained by differentiating (5.18) with respect to r:

$$\partial_t\xi - \partial_r^2(|\xi|^{p-2}\xi) - \frac{n-1}{r}\partial_r(|\xi|^{p-2}\xi) + \frac{n-1}{r^2}|\xi|^{p-2}\xi + (p-1)|\xi|^{p-3}\xi\partial_r\xi - \xi = 0,$$

which is a parabolic equation (of porous medium type) and satisfies a maximum principle. Since 0 is a solution to the above equation, the derivative $\xi = \partial_r v$ remains nonpositive if it is initially nonpositive and it follows that v is radially non-increasing. □

We are now in position to end the proof of Theorem 5.2 for radially non-increasing initial data. More precisely, we have the following upper and lower bounds for the edge $\varrho(\tau)$ defined in (5.28) of the support of $v(\tau)$.

Lemma 5.6. *Let $u_0 = u_0(r)$ be a radially non-increasing function satisfying* (5.3) *and denote the solution of* (5.18) *with initial condition u_0 by v. For any $c \in (1/2,1)$, there exists $\tau_1(c) > 0$ such that, for any $\tau \geq \tau_1(c)$, we have:*

$$1 + c(\tau - \tau_1(c)) \leq \varrho(\tau) \leq R_0 + \frac{p-1}{p-2}\ \|u_0\|_\infty^{(p-2)/(p-1)} + \tau\,. \tag{5.40}$$

In particular, we obtain that $\varrho(\tau)/\tau \to 1$ as $\tau \to \infty$.

Proof. The upper bound follows by comparison with the explicit traveling wave solutions (5.33). More precisely, we define

$$R_1 := R_0 + \frac{p-1}{p-2}\ \|u_0\|_\infty^{(p-2)/(p-1)} \tag{5.41}$$

and consider the function $\overline{v}(x,\tau) = f_{1,R_1}(x_1-\tau)$, which is a solution of (5.18) by Lemma 5.3. If $x = (x_i)_{1\leq i\leq n} \in \mathbb{R}^n$ is such that $x_1 \geq R_0$, then $|x| \geq R_0$ and $u_0(x) = 0 \leq \overline{v}(x,0)$ while, if $x_1 \leq R_0$,

$$\begin{aligned} u_0(x) &\leq \|u_0\|_\infty \leq \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)}\ (R_1-R_0)^{(p-1)/(p-2)} \\ &\leq \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)}\ (R_1-x_1)^{(p-1)/(p-2)} = \overline{v}(x,0)\,. \end{aligned}$$

The comparison principle then entails that $v(x,\tau) \le \overline{v}(x,\tau)$ for $(x,\tau) \in \mathbb{R}^n \times [0,\infty)$, from which we conclude that $\mathcal{P}_v(\tau) \subseteq \{x \in \mathbb{R}^n \ : \ x_1 \le R_1 + \tau\}$. Owing to the rotational invariance of (5.18), we actually have $\mathcal{P}_v(\tau) \subseteq \{x \in \mathbb{R}^n \ : \ \langle x, \omega\rangle \le R_1 + \tau\}$ for every $\omega \in \mathbb{S}^{n-1}$ and $\tau \ge 0$, and thus

$$\mathcal{P}_v(\tau) \subseteq B(0, R_1 + \tau) . \tag{5.42}$$

The lower bound follows from comparison with the subsolutions constructed in (5.39). Fix $c \in (1/2, 1)$ and put $r_1 := 1 + c\tau_0(c)$, $\tau_0(c)$ being defined by (5.38). Since $v(\tau)$ is radially non-increasing for all $\tau \ge 0$ by Lemma 5.5, we infer from Proposition 5.2 that, for $x \in B(0, r_1)$ and $\tau \ge T_{r_1}$,

$$v(x,\tau) \ge v\left(\frac{r_1 x}{|x|}, \tau\right) \ge \varepsilon_{r_1}\ \tau^{(p-1)/(p-2)} .$$

Define $\tau_1(c)$ by

$$\tau_1(c) := \max\left\{\tau_0(c), T_{r_1}, \left(\frac{M_{c,(1-c)/(1+c)}}{\varepsilon_{r_1}}\right)^{(p-2)/(p-1)}\right\},$$

so that the previous inequality and the properties of $v_{c,1}$ defined in (5.39) guarantee that

$$v(x, \tau_1(c)) \ge M_{c,(1-c)/(1+c)} \ge v_{c,1}(x, \tau_0(c)) , \quad x \in B(0, r_1) .$$

Since $v_{c,1}(x, \tau_0(c)) = 0$ for $x \notin B(0, r_1)$, we also have $v(x, \tau_1(c)) \ge v_{c,1}(x, \tau_0(c))$ for $x \notin B(0, r_1)$. Recalling that $v_{c,1}$ is a subsolution to (5.18) in $\mathbb{R}^n \times (\tau_0(c), \infty)$, we infer from the comparison principle that

$$v(x, \tau + \tau_1(c)) \ge v_{c,1}(x, \tau + \tau_0(c)), \quad (x,\tau) \in Q . \tag{5.43}$$

Consequently, $v(x, \tau + \tau_1(c)) > 0$ if $x \in B(0, r_1 + c\tau)$, whence

$$B(0, 1 + c(\tau + \tau_0(c) - \tau_1(c))) \subset \mathcal{P}_v(t) , \quad \tau \ge \tau_1(c) . \tag{5.44}$$

This readily implies that

$$\varrho(\tau) \ge 1 + c(\tau + \tau_0(c) - \tau_1(c)) \ge 1 + c(\tau - \tau_1(c)) , \quad \tau \ge \tau_1(c) .$$

In particular, we deduce from (5.42) and (5.44) that

$$\liminf_{\tau\to\infty} \frac{\varrho(\tau)}{\tau} \ge c \quad \text{for any} \quad c \in (1/2, 1) \quad \text{and} \quad \limsup_{\tau\to\infty} \frac{\varrho(\tau)}{\tau} \le 1 ,$$

which implies that $\varrho(\tau)/\tau \to 1$ as $\tau \to \infty$. □

Rephrasing the rescaling and coming back to the notation with $t = (e^{(p-2)\tau} - 1)/(p-2)$ and $\gamma(t) = \varrho(\tau)$, we find the result of Theorem 5.2 for radially non-increasing inital data. The extension to arbitrary initial data satisfying (5.3) is performed in Section 5.5. Moreover, we notice that the speed is the same in any direction $\omega \in \mathbb{S}^{n-1}$, as stated.

5.4 Proof of Theorem 5.1

5.4.1 Scaling variables II

According to Proposition 5.2, as $\tau \to \infty$ the solution v to (5.18), (5.19) expands in space and grows unboundedly in time. In order to take into account such phenomena, we introduce next a further scaling of the space variable

$$y := \frac{x}{1+\tau}\,, \tag{5.45}$$

together with the new unknown function $w = w(y,\tau)$ defined by

$$v(x,\tau) = (1+\tau)^{(p-1)/(p-2)}\ w\left(\frac{x}{1+\tau},\tau\right)\,, \quad (x,\tau) \in \mathbb{R}^n \times [0,\infty)\,. \tag{5.46}$$

It follows from (5.18) and (5.19) that w solves

$$\partial_\tau w - \frac{1}{1+\tau}\left(\Delta_p w + y\cdot\nabla w - \frac{p-1}{p-2}\,w\right) + |\nabla w|^{p-1} - w = 0\ \ , \quad (y,\tau) \in Q\,, \tag{5.47}$$

with the same initial condition

$$w(0) = u_0\,, \quad y \in \mathbb{R}^n\,. \tag{5.48}$$

Throughout this section we assume that u_0 is radially non-increasing besides (5.3). In particular, $u_0(0) > 0$. We gather several properties of w in the next lemma.

Lemma 5.7. *There is a positive constant C_4 depending only on p, n, and u_0 such that*

$$\|w(\tau)\|_1 + \|w(\tau)\|_\infty + \|\nabla w(\tau)\|_\infty \le C_4\,, \quad \tau \ge 0\,, \tag{5.49}$$

$$w(y,\tau) \ge \frac{1}{C_4}\left(r_*^2 - |y|^2\right)_+^{(p-1)/(p-2)}\ , \quad (y,\tau) \in Q\,, \tag{5.50}$$

the radius r_* being defined in Corollary 5.1. Moreover,

$$\mathcal{P}_w(\tau) := \{y \in \mathbb{R}^n :\ \ w(y,\tau) > 0\} \subseteq B\left(0, 1 + \frac{R_1}{1+\tau}\right) \tag{5.51}$$

for $\tau \ge 0$ where R_1 is defined by (5.41). In addition, for any $c \in (1/2,1)$, we have

$$B\left(0, c - \frac{\tau_1(c)}{1+\tau}\right) \subset \mathcal{P}_w(\tau) \quad \text{for } \tau \ge \tau_1(c)\,, \tag{5.52}$$

the time $\tau_1(c) > 0$ being defined in Lemma 5.6.

Proof. The estimates (5.49) and (5.50) readily follow from (5.20) and (5.25), while (5.51) is a consequence of (5.42). The assertion about the ball $B(0, c - \tau_1(c)/(1+\tau))$ follows from (5.44). $\square$

At this point, (5.47) indicates that $w(\tau)$ behaves as $\tau \to \infty$ as the solution $\tilde{w}$ to the Hamilton-Jacobi equation $\partial_\tau \tilde{w} + |\nabla \tilde{w}|^{p-1} - \tilde{w} = 0$ in Q which is known to converge to a stationary solution uniquely determined by the limit of the support of $\tilde{w}(\tau)$ as $\tau \to \infty$, see, e.g., [100, Theorem A.2]. As an intermediate step, we thus have to identify the limit of the support of $w(\tau)$ as $\tau \to \infty$. Thanks to (5.51), we already know that it is included in $B(0,1)$ but the information in (5.52) is yet too weak to exclude the vanishing of $W(\tau)$ outside a ball of radius smaller than one. To complete the proof of Theorem 5.1 for radially non-increasing initial data, we show first that the asymptotic limit is supported exactly in the ball $B(0,1)$. Then we use a viscosity technique, the same that has been used in the previous paper [99] to establish the convergence to the expected stationary solution.

5.4.2 Proof of Theorem 5.1: $n = 1$

We first consider the one-dimensional case $n = 1$ and divide the proof into several technical steps.

Step 1. A special family of subsolutions. Given $c \in (1/2, 1)$, we have

$$v(x,\tau) \geq v_{c,1}(x, \tau + \tau_0(c) - \tau_1(c)), \quad (x,\tau) \in \mathbb{R} \times [\tau_1(c), \infty),$$

by (5.43), the times $\tau_0(c)$ and $\tau_1(c)$ being defined in (5.38) and Lemma 5.6, respectively. Then,

$$w(y,\tau) \geq w_c(y,\tau) := \frac{1}{(1+\tau)^{(p-1)/(p-2)}}\, v_{c,1}(y(1+\tau), \tau + \tau_0(c) - \tau_1(c)) \tag{5.53}$$

for $(y,\tau) \in \mathbb{R} \times [\tau_1(c), \infty)$.

Step 2. An explicit family of supersolutions. Let us introduce the following family of functions:

$$F_R(y,\tau) = \left(\frac{p-2}{p-1}\right)^{(p-1)/p-2} \left(\frac{\tau+R}{\tau+1} - |y|\right)_+^{(p-1)/(p-2)}, \quad (y,\tau) \in Q\,. \tag{5.54}$$

We easily obtain by direct calculation that F_R is a classical solution of (5.47) for $y \neq 0$, and for all parameter values $R \geq 0$. However, near $y = 0$, it is only a supersolution both in the weak and the viscosity sense. The latter is straightforward to verify using the definition of viscosity subsolutions and supersolutions with jets, as in the classical survey [47]. Let us mention at this point that these functions can be used in a comparison argument to give an alternative proof of (5.51).

Remark 2. *This family of functions arises naturally if we think about asymptotics. Indeed, as already mentioned, we formally expect that the asymptotic profiles of* (5.47) *should be given by solutions of the stationary Hamilton-Jacobi equation*

$$|\nabla \tilde{w}|^{p-1} - \tilde{w} = 0, \tag{5.55}$$

supported in some ball $B(0,R)$, that is

$$H_R(y) := \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (R-|y|)_+^{(p-1)/(p-2)} , \quad y \in \mathbb{R} .$$

Making then the "ansatz" that, for large times, the solution of (5.47) *should behave in a similar way as its limit, we write*

$$w(y,\tau) \sim \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (C(\tau)-|y|)_+^{(p-1)/(p-2)} .$$

Integrating the resulting ordinary differential equation for $C(\tau)$, we arrive at the family of explicit exact profiles F_R given by (5.54).

Step 3. Constructing suitable subsolutions. We now face the problem of finding suitable subsolutions with similar behaviour. Since the F_R's are classical solutions to (5.47) except at $y=0$, we expect to be able to construct also a family of subsolutions based on them. To this end, we consider the "damped" family $F_{R,\vartheta,\beta}$ defined by

$$F_{R,\vartheta,\beta}(y,\tau) := \vartheta \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} \left(\frac{\beta(\tau+R)}{\tau+1} - |y|\right)_+^{(p-1)/(p-2)} , \quad (y,\tau) \in Q , \tag{5.56}$$

for parameters $R \in (0,1)$, $\vartheta \in (0,1]$, and $\beta \in (1/2,1]$. Observe that, since $(p-1)/(p-2) > 1$, $F_{R,\vartheta,\beta}$ and $|\nabla F_{R,\vartheta,\beta}|^{p-2}\nabla F_{R,\vartheta,\beta}$ both belong to $\mathcal{C}^1((\mathbb{R}\setminus\{0\})\times[0,\infty))$. For $\vartheta \in (0,1)$, $\beta \in (1/2,1]$, $\tau > 0$ and $y \neq 0$, we calculate

$$\begin{aligned}
\partial_\tau F_{R,\vartheta,\beta} &- \frac{1}{1+\tau}\left(\Delta_p F_{R,\vartheta,\beta} + y\cdot\nabla F_{R,\vartheta,\beta} - \frac{p-1}{p-2}F_{R,\vartheta,\beta}\right) + |\nabla F_{R,\vartheta,\beta}|^{p-1} - F_{R,\vartheta,\beta} \\
&= \vartheta\beta\frac{1-R}{(1+\tau)^2}F_{R,1,\beta}^{1/(p-1)} - \frac{\vartheta}{1+\tau}\left(\vartheta^{p-2} - \frac{\beta(\tau+R)}{\tau+1}\right) F_{R,1,\beta}^{1/(p-1)} - \vartheta(1-\vartheta^{p-2})F_{R,1,\beta} \\
&= \vartheta\left(\frac{\beta-\vartheta^{p-2}}{1+\tau} - (1-\vartheta^{p-2})F_{R,1,\beta}^{(p-2)/(p-1)}\right) F_{R,1,\beta}^{1/(p-1)} \\
&\le \vartheta(1-\vartheta^{p-2})F_{R,1,\beta}^{1/(p-1)}\left[\frac{1}{1+\tau} - \frac{p-2}{p-1}\left(\frac{\beta(\tau+R)}{\tau+1} - |y|\right)\right] .
\end{aligned}$$

Analyzing the sign of the last expression and taking into account that $\vartheta \in (0,1)$, we obtain that $F_{R,\vartheta,\beta}$ has the following properties:

$$\begin{aligned}
&F_{R,\vartheta,\beta} \text{ is a classical subsolution to (5.47) in} \\
&\{(y,\tau)\in Q \ : \ \tau \ge \tau_2(R,\beta) ,\ 0 < |y| \le K_{R,\beta}(\tau)\}
\end{aligned} \tag{5.57}$$

with

$$\tau_2(R,\beta) := \frac{p-1}{\beta(p-2)} - R \quad \text{and} \quad K_{R,\beta}(\tau) := \frac{\beta(\tau+R)}{\tau+1} - \frac{p-1}{p-2}\,\frac{1}{\tau+1} , \tag{5.58}$$

and

$$F_{R,\vartheta,\beta} \text{ vanishes for } |y| \geq \frac{\beta(\tau+R)}{\tau+1} \text{ and } \tau \geq 0\,. \tag{5.59}$$

Let us notice here that both the edge of the support of $F_{R,\vartheta,\beta}$ and the constant $K_{R,\beta}(\tau)$, where the behaviour changes, do not depend on ϑ. While the two properties (5.57) and (5.59) are suitable for our purpose, the function $F_{R,\vartheta,\beta}$ does not behave in a suitable way near $y=0$ (where it is a viscosity supersolution) and in an asymptotically small region near the edge of its support (where it is a classical supersolution). However, we already have a positive bound from below for w in a small neighbourhood of $y=0$ by (5.50) which allows us to remedy the first bad property of $F_{R,\vartheta,\beta}$. More precisely, we infer from (5.50) that

$$w(y,\tau) \geq C_5 := \frac{1}{C_4}\left(\frac{3r_*^2}{4}\right)^{(p-1)/(p-2)} > 0\,, \quad (y,\tau) \in B(0,r_*/2) \times [0,\infty)\,,$$

whence

$$w(y,\tau) \geq \vartheta \geq F_{R,\vartheta,\beta}(y,\tau)\,, \quad (y,\tau) \in B(0,r_*/2) \times [0,\infty)\,, \tag{5.60}$$

provided that

$$0 < \vartheta < \min\{1, C_5\}\,. \tag{5.61}$$

Consider next

$$\tau \geq \tau_2(R,\beta) \quad \text{and} \quad K_{R,\beta}(\tau) \leq |y| \leq \frac{\beta(\tau+R)}{\tau+1}\,.$$

Then

$$\begin{aligned} F_{R,\vartheta,\beta}(y,\tau) &\leq \vartheta \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} \left(\frac{p-1}{p-2}\,\frac{1}{1+\tau}\right)^{(p-1)/(p-2)} \\ &= \frac{\vartheta}{(1+\tau)^{(p-1)/(p-2)}}\,. \end{aligned} \tag{5.62}$$

Now, if $c \in (\beta,1)$, we have

$$|y|(1+\tau) \leq \beta(\tau+R) \leq \tilde{z}_{c,1,(1-c)/(1+c)} + c(\tau+\tau_0(c)-\tau_1(c))$$

as soon as

$$\tau \geq \tau_3(c,R,\beta) := \frac{\beta R + c(\tau_1(c)-\tau_0(c)) - \tilde{z}_{c,1,(1-c)/(1+c)}}{c-\beta}\,. \tag{5.63}$$

In that case,

$$w_c(y,\tau) = \frac{1}{(1+\tau)^{(p-1)/(p-2)}}\, v_{c,1}(\tau+\tau_0(c)-\tau_1(c), y(1+\tau)) = \frac{M_{c,(1-c)/(1+c)}}{(1+\tau)^{(p-1)/(p-2)}}$$

according to the properties (5.39) of $v_{c,1}$. Recalling (5.53) and (5.62) we realize that

$$F_{R,\vartheta,\beta}(y,\tau) \leq w_c(y,\tau) \leq w(y,\tau)\,, \quad K_{R,\beta}(\tau) \leq |y| \leq \frac{\beta(\tau+R)}{\tau+1}\,, \tag{5.64}$$

provided

$$c \in (\beta, 1)\,, \quad \vartheta < \min\left\{1, M_{c,(1-c)/(1+c)}\right\}\,, \quad \tau \ge \max\left\{\tau_1(c), \tau_2(R,\beta), \tau_3(c,R,\beta)\right\}\,. \tag{5.65}$$

After this preparation, we are in a position to establish a positive lower bound for w on the ball $B(0,1-\varepsilon)$ for any $\varepsilon \in (0,1/4)$. Indeed, we fix $\varepsilon \in (0,1/4)$, choose $c = 1-\varepsilon$, $R = \beta = 1-2\varepsilon$, and define

$$\tau_4(\varepsilon) := \max\left\{\frac{\tau_1(1-\varepsilon)}{\varepsilon}, \tau_2(1-2\varepsilon, 1-2\varepsilon), \tau_3(1-\varepsilon, 1-2\varepsilon, 1-2\varepsilon)\right\}.$$

As $\tau_4(\varepsilon) > \tau_1(1-\varepsilon)/\varepsilon$, (5.52) guarantees that $B(0,1-2\varepsilon) \subset \mathcal{P}_w(\tau_4(\varepsilon))$ and there is thus $m_\varepsilon \in (0,1)$ such that

$$w(y, \tau_4(\varepsilon)) \ge m_\varepsilon\,, \quad y \in B(0, 1-2\varepsilon)\,. \tag{5.66}$$

Now, for $\vartheta \in (0,1)$ satisfying

$$0 < \vartheta < \min\left\{m_\varepsilon, C_5, M_{1-\varepsilon, \varepsilon/(2-\varepsilon)}\right\} \tag{5.67}$$

we infer from (5.58), (5.60), (5.61), (5.64), (5.65), and (5.66) that

$$F_{1-2\varepsilon,\vartheta,1-2\varepsilon}(y,\tau) \le w(y,\tau)\,, \quad |y| \in \left\{\frac{r_*}{2}, K_{1-2\varepsilon,1-2\varepsilon}(\tau)\right\}\,, \quad \tau \ge \tau_4(\varepsilon)\,,$$

and

$$F_{1-2\varepsilon,\vartheta,1-2\varepsilon}(y,\tau_4(\varepsilon)) \le \vartheta \le m_\varepsilon \le w(y,\tau_4(\varepsilon))\,, \quad \frac{r_*}{2} \le |y| \le K_{1-2\varepsilon,1-2\varepsilon}(\tau_4(\varepsilon)) \le 1-2\varepsilon\,.$$

It then follows from (5.47), (5.57), and the comparison principle that

$$F_{1-2\varepsilon,\vartheta,1-2\varepsilon}(y,\tau) \le w(y,\tau)\,, \quad \frac{r_*}{2} \le |y| \le K_{1-2\varepsilon,1-2\varepsilon}(\tau)\,, \quad \tau \ge \tau_4(\varepsilon)\,.$$

Recalling (5.59), (5.60), and (5.64), we have thus established that

$$F_{1-2\varepsilon,\vartheta,1-2\varepsilon}(y,\tau) \le w(y,\tau)\,, \quad (y,\tau) \in \mathbb{R} \times [\tau_4(\varepsilon), \infty)\,, \tag{5.68}$$

for all $\vartheta \in (0,1)$ satisfying (5.67).

Step 4. Positive bound from below. For $\varepsilon \in (0,1/4)$, fix $\vartheta_\varepsilon \in (0,1)$ satisfying (5.67). According to (5.68), we have, for $\tau \ge \tau_4(\varepsilon) + 1$ and $y \in B(0, 1-3\varepsilon)$,

$$\begin{aligned} w(y,\tau) \quad &\ge \quad \vartheta_\varepsilon \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} \left(\frac{(1-2\varepsilon)(\tau+1-2\varepsilon)}{\tau+1} - |y|\right)_+^{(p-1)/(p-2)} \\ &\ge \quad \vartheta_\varepsilon \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} \left(\frac{\varepsilon(\tau-1+4\varepsilon)}{\tau+1}\right)_+^{(p-1)/(p-2)} \\ &\ge \quad \mu_\varepsilon := \vartheta_\varepsilon \left(\frac{2(p-2)\varepsilon^2}{p-1}\right)^{(p-1)/(p-2)} > 0\,. \end{aligned}$$

We have thus proved that, for all $\varepsilon \in (0,1/4)$, there are $\mu_\varepsilon > 0$ and $\tau_5(\varepsilon) := \tau_4(\varepsilon)+1$ such that

$$0 < \mu_\varepsilon \le w(y,\tau)\,, \quad (y,\tau) \in B(0,1-3\varepsilon) \times [\tau_5(\varepsilon),\infty)\,. \tag{5.69}$$

Step 5. Convergence. Viscosity argument. To complete the proof, we use an argument relying on the theory of viscosity solutions in a similar way as in the paper [99] for the subcritical case of (5.1) with $q \in (1,p-1)$. We thus employ the technique of half-relaxed limits [16] in the same fashion as in [116, Section 3] and [99]. To this end, we pass to the logarithmic time and introduce the new variable $s := \log(1+\tau)$ along with the new unknown function

$$w(y,\tau) = \omega(y, \log(1+\tau))\,, \quad (y,\tau) \in \mathbb{R} \times [0,\infty)\,.$$

Then, $\partial_\tau w(y,\tau) = e^{-s}\partial_s\omega(y,s)$ and it follows from (5.47) and (5.48) that ω solves

$$e^{-s}\left(\partial_s\omega - \Delta_p\omega - y\cdot\nabla\omega + \frac{p-1}{p-2}\,\omega\right) + |\nabla\omega|^{p-1} - \omega = 0 \ , \quad (y,s) \in Q\,, \tag{5.70}$$

with initial condition $\omega(0) = u_0$. We readily infer from Lemma 5.7 that

$$\|\omega(s)\|_1 + \|\omega(s)\|_\infty + \|\nabla\omega(s)\|_\infty \le C_4\,, \quad s \ge 0\,, \tag{5.71}$$

$$\omega(y,s) = 0 \quad \text{for} \quad s \ge 0 \quad \text{and} \quad |y| \ge 1 + R_1 e^{-s}\,. \tag{5.72}$$

We next introduce the half-relaxed limits

$$\omega_*(y) := \liminf_{(\sigma,z,\lambda)\to(\sigma,y,\infty)} \omega(z,\lambda+\sigma) \quad \text{and} \quad \omega^*(y) := \limsup_{(\sigma,z,\lambda)\to(\sigma,y,\infty)} \omega(z,\lambda+\sigma),$$

for $(y,s) \in Q$, which are well-defined according to the uniform bounds in (5.71) and indeed do not depend on $s > 0$. Then, the definition of ω_* and ω^* clearly ensures that

$$0 \le \omega_*(y) \le \omega^*(y) \quad \text{for} \quad y \in \mathbb{R}\,, \tag{5.73}$$

while the uniform bounds (5.71) and the Rademacher theorem warrant that ω_* and ω^* both belong to $W^{1,\infty}(\mathbb{R})$. Finally, by Proposition 5.3 in the Appendix, applied to (5.70), ω_* and ω^* are viscosity supersolution and subsolution, respectively, to the Hamilton-Jacobi equation

$$H(\zeta,\nabla\zeta) := |\nabla\zeta|^{p-1} - \zeta = 0 \quad \text{in} \quad \mathbb{R}\,. \tag{5.74}$$

Our aim is now to show that $\omega_* \ge \omega^*$ in $\mathbb{R}$ (which implies that $\omega_* = \omega^*$ by (5.73)). Since ω^* and ω_* are subsolution and supersolution to (5.74), respectively, such an inequality would follow from a comparison principle which cannot be applied yet without further information on ω^* and ω_*. We actually need to prove the following two facts:

(a) $\omega_*(y) = \omega^*(y) = 0$ if $|y| \ge 1$,

(b) $\omega^*(y) \ge \omega_*(y) > 0$ if $y \in B(0,1)$, and then to follow the technique used in [99] to conclude that $\omega_* = \omega^*$ and identify the limit.

To prove assertion (a), let us take $y \in \mathbb{R}$ with $|y| > 1$. We then deduce from (5.72) that there exists $s_1(y) > 0$ such that $\omega(y,s) = 0$ for $s \geq s_1(y)$. Pick sequences $(\sigma_n)_{n\geq 1}$, $(\lambda_n)_{n\geq 1}$, and $(z_n)_{n\geq 1}$ such that $\sigma_n \to 0$, $\lambda_n \to \infty$, $z_n \to y$, and $\omega(z_n, \sigma_n + \lambda_n) \to \omega^*(y)$. On the one hand, there exists $n_1(y) > 0$ such that $\sigma_n + \lambda_n > s_1(y)$ for any $n \geq n_1(y)$; hence $\omega(y, \sigma_n + \lambda_n) = 0$ for any $n \geq n_1(y)$. On the other hand, we can write:

$$|\omega(z_n, \sigma_n + \lambda_n) - \omega(y, \sigma_n + \lambda_n)| \leq |y - z_n| \|\nabla \omega(\sigma_n + \lambda_n)\|_\infty \leq C_4 |y - z_n| \to 0,$$

hence $\omega^*(y) = 0 = \omega_*(y)$ for any $y \in \mathbb{R}$ with $|y| > 1$. In addition, since ω^* and ω_* are continuous, it follows that $\omega^* = \omega_* = 0$ also for $|y| = 1$, hence assertion (a).

To prove assertion (b), let us take $y \in B(0,1)$. Then, there exists $\varepsilon \in (0, 1/4)$ such that $y \in B(0, 1-4\varepsilon)$. Since $1 - 3\varepsilon > 1 - 4\varepsilon$, there is $r_2(y) > 0$ such that $B(y, r_2(y)) \subset B(0, 1-3\varepsilon)$ and we deduce from (5.69) that there exists $s_2(\varepsilon) := \log(\tau_5(\varepsilon) + 1) > 0$ such that $\omega(z,s) \geq \mu_\varepsilon$ for any $s \geq s_2(\varepsilon)$ and $z \in B(y, r_2(y))$. We now pick sequences $(\sigma_n)_{n\geq 1}$, $(\lambda_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ such that $\sigma_n \to 0$, $\lambda_n \to \infty$, $z_n \to y$, and $\omega(z_n, \sigma_n + \lambda_n) \to \omega_*(y)$. Then there exists again $n_2(y) > 0$ such that $\sigma_n + \lambda_n > s_2(y)$ and $z_n \in B(y, r_2(y))$ for any $n \geq n_2(y)$. Consequently $\omega(z_n, \sigma_n + \lambda_n) \geq \mu_\varepsilon$ for any $n \geq n_2(y)$. This readily implies that $\omega^*(y) \geq \omega_*(y) \geq \mu_\varepsilon > 0$, hence (b) is proved.

We follow the lines of [99] and introduce

$$W_*(y) = \frac{p-1}{p-2} \omega_*(y)^{(p-2)/(p-1)}, \quad W^*(y) = \frac{p-1}{p-2} \omega^*(y)^{(p-2)/(p-1)}, \tag{5.75}$$

for any $y \in B(0,1)$. From Proposition 5.4, it follows that W_* and W^* are respectively viscosity supersolution and subsolution of the eikonal equation

$$|\nabla \zeta| = 1 \quad \text{in } B(0,1),$$

with boundary conditions $W^*(y) = W_*(y) = 0$ for $|y| = 1$ and are both positive in $B(0,1)$. Using the comparison principle of Ishii [85], we find that $W^*(y) \leq W_*(y)$, hence they should be equal by (5.73). It follows that $\omega_* = \omega^* = W$ in $B(0,1)$, where W is the viscosity solution to (5.13)

$$|\nabla W|^{p-1} - W = 0 \quad \text{in} \quad B(0,1)\,, \qquad W = 0 \quad \text{on} \quad \partial B(0,1)\,,$$

which is actually explicit and given by

$$W(x) := \left(\frac{p-2}{p-1} \, (1 - |x|)_+ \right)^{(p-1)/(p-2)},$$

as stated in Theorem 5.1. In addition, the equality $\omega_* = \omega^*$ and (5.72) entail the convergence of $\omega(s)$ as $s \to \infty$ towards W in $L^\infty(\mathbb{R})$ by Lemma 4.1 in [17] or Lemma V.1.9 in [10]. We end the proof by rephrasing the two scaling steps and arriving in this way to (5.10). $\square$

5.4.3 Proof of Theorem 5.1: $n \geq 2$

We now prove Theorem 5.1 for radially non-increasing initial data to the problem posed in dimension $n \geq 2$. We follow the same steps as in dimension $n = 1$, and we only indicate below the main differences that appear. These differences are mainly given by the appearance of the new term

$$\frac{n-1}{r} |\partial_r w|^{p-2} \partial_r w \,, \quad r = |y| \,, \tag{5.76}$$

in the radial form of the p-Laplacian term. As we shall see, performing carefully the same steps as for dimension $n = 1$, we find that this term does not change anything in an essential way. We follow the same division into steps as the case $n = 1$.

Step 1. Thanks to the construction performed in Section 5.3.2, this step is the same as in dimension $n = 1$.

Step 2. Due to the appearance of the extra term (5.76) in the radial form of the equation (5.47), we check by direct calculation that, in dimension $n \geq 2$, the function F_R given by formula (5.54) is now a strict supersolution to (5.47) in Q. Indeed, for $y \neq 0$,

$$\partial_\tau F_R - \frac{1}{1+\tau} \left(\Delta_p F_R + y \cdot \nabla F_R - \frac{p-1}{p-2} F_R \right) + |\nabla F_R|^{p-1} - F_R = \frac{n-1}{(1+\tau)|y|} F_R \,.$$

Moreover, its singularity at $y = 0$ is now stronger. This seems to introduce a new difficulty, but we will see that it can be handled by the same perturbation techniques. Let us notice at this moment that F_R can be used for upper bounds in the same way as in the case $n = 1$, and that F_R still solves the limit Hamilton-Jacobi equation (5.55).

Step 3. In order to construct subsolutions starting from the family of functions F_R, we follow again the ideas of the case $n = 1$. The calculations will be different in some points. We consider again the damped family $F_{R,\vartheta,\beta}$ defined in (5.56) for $R \in (0,1)$, $\vartheta \in (0,1)$, and $\beta \in (1/2, 1]$. For $y \neq 0$ we have

$$\begin{aligned} Y &:= \partial_\tau F_{R,\vartheta,\beta} - \frac{1}{1+\tau} \left(\Delta_p F_{R,\vartheta,\beta} + y \cdot \nabla F_{R,\vartheta,\beta} - \frac{p-1}{p-2} F_{R,\vartheta,\beta} \right) + |\nabla F_{R,\vartheta,\beta}|^{p-1} - F_{R,\vartheta,\beta} \\ &= \vartheta F_{R,1,\beta}^{1/(p-1)} \left[\frac{\beta - \vartheta^{p-2}}{1+\tau} + \frac{(n-1)\vartheta^{p-2}}{(1+\tau)|y|} F_{R,1,\beta}^{(p-2)/(p-1)} - (1-\vartheta^{p-2}) F_{R,1,\beta}^{(p-2)/(p-1)} \right] . \end{aligned}$$

At this point, we further assume that $|y| > r_*/2$, the radius r_* being defined in Corollary 5.1, and that

$$\vartheta^{p-2} \leq \frac{(1-\beta) r_*}{2(n-1)} \,. \tag{5.77}$$

Since $F_{R,1,\beta} \leq 1$, we obtain

$$\begin{aligned} Y &\leq \vartheta F_{R,1,\beta}^{1/(p-1)} \left[\frac{\beta - \vartheta^{p-2}}{1+\tau} + \frac{2(n-1)\vartheta^{p-2}}{(1+\tau) r_*} - (1-\vartheta^{p-2}) F_{R,1,\beta}^{(p-2)/(p-1)} \right] \\ &\leq \vartheta (1-\vartheta^{p-2}) F_{R,1,\beta}^{1/(p-1)} \left[\frac{1}{1+\tau} - \frac{p-2}{p-1} \left(\frac{\beta(\tau+R)}{\tau+1} - |y| \right) \right] , \end{aligned}$$

from which we conclude that

$$\begin{array}{l} F_{R,\vartheta,\beta} \text{ is a classical subsolution to (5.47) in} \\ \{(y,\tau)\in Q \ : \ \tau\ge\tau_2(R,\beta)\,,\ (r_*/2)<|y|\le K_{R,\beta}(\tau)\}\,, \end{array} \tag{5.78}$$

where $\tau_2(R,\beta)$ and $K_{R,\beta}(\tau)$ are still given by (5.58). We now proceed as in the one dimensional case to establish (5.68) for all $\vartheta\in(0,1)$ satisfying (5.67) along with

$$\vartheta^{p-2}\le\frac{\varepsilon r_*}{N-1}\,,$$

for (5.77) to be satisfied.

Steps 4 & 5. The final steps of the proof are similar to the one dimensional case. □

5.5 Arbitrary initial data

So far, we have proved Theorems 5.1 and 5.2 for radially non-increasing initial data satisfying (5.3). We now extend these two results to general initial data satisfying (5.3).

Proof of Theorems 5.1 and 5.2. Since $u_0\not\equiv 0$, there are $x_0\in\mathbb{R}^n$, $r_0>0$, and $\eta_0>0$ such that $u_0(x)\ge 2\eta_0$ for $x\in B(x_0,r_0)$. Then, there exists a radially non-increasing initial condition $\tilde u_0$ satisfying (5.3) but with support in $B(0,r_0)$ and such that $\|\tilde u_0\|_\infty\le\eta_0$ and $\tilde u_0(x)\le u_0(x-x_0)$ for $x\in\mathbb{R}^n$. Similarly, there is a radially non-increasing initial condition $\tilde U_0$ satisfying (5.3) but with support in $B(0,\tilde R_0)$ for some $\tilde R_0>R_0$ and such that $\tilde U_0(x)\ge\|u_0\|_\infty$ for $x\in B(0,R_0)$. Denoting the solutions to (5.1) by $\tilde u$ and $\tilde U$ with initial conditions $\tilde u_0$ and $\tilde U_0$, respectively, the comparison principle and the translational invariance of (5.1) ensure that

$$\tilde u(x+x_0,t)\le u(x,t)\le\tilde U(x,t)\,,\quad (x,t)\in Q\,. \tag{5.79}$$

Moreover, since

$$\left|\left(1-\frac{(p-2)|x+x_0|}{\log t}\right)_+^{(p-1)/(p-2)}-\left(1-\frac{(p-2)|x|}{\log t}\right)_+^{(p-1)/(p-2)}\right|\le\frac{(p-1)|x_0|}{\log t}\,,$$

and Theorems 5.1 and 5.2 apply to both $\tilde u$ and $\tilde U$, the expected results follow from (5.79). □

Appendix. Some results about viscosity solutions

We state, for the sake of completeness, some standard results in the theory of viscosity solutions, that we use in the proof of Theorem 5.1. The first one concerns the "viscosity" limit of a family of small perturbations and can be found in [17, Theorem 4.1].

Proposition 5.3. *Let u_ε be a viscosity subsolution (resp. a viscosity supersolution) of the equation*

$$H_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon,D^2u_\varepsilon)=0\quad\text{in }\mathbb{R}^n,$$

where H_ε is uniformly bounded in all variables and degenerate elliptic. Suppose that $\{u_\varepsilon\}$ is a uniformly bounded family of functions. Then

$$u^*(x) := \limsup_{(y,\varepsilon)\to(x,0)} u_\varepsilon(y) \tag{5.80}$$

is a subsolution of the equation

$$H_*(x, u, \nabla u, D^2 u) = 0, \tag{5.81}$$

In the same way,

$$u_*(x) := \liminf_{(y,\varepsilon)\to(x,0)} u_\varepsilon(y)$$

is a supersolution of $H^*(x, u, \nabla u, D^2 u) = 0$. Here, H_* and H^* are constructed in the same way as u_* and u^*.

In other words, this result can be applied to asymptotically small perturbations of a known equation, as we do in Section 5.4.

We also use the following result:

Proposition 5.4. *Let $u \in C(\Omega)$ be a viscosity solution of*

$$H(x, u, \nabla u) = 0 \quad \text{in } \Omega, \tag{5.82}$$

where $\Omega \subset \mathbb{R}^n$ and H is a continuous function. If $\Phi \in C^1(\mathbb{R})$ is an increasing function, then $v = \Phi(u)$ is a viscosity solution of

$$H\left(x, \Phi^{-1}(v(x)), (\Phi^{-1})'(v(x))\nabla v(x)\right) = 0. \tag{5.83}$$

The same result holds true for subsolutions and supersolutions and can be found in [17]. In particular, we use this result in order to pass from the Hamilton-Jacobi equation $|\nabla u|^{p-1} - u = 0$ to the standard eikonal equation $|\nabla v| = 1$. Finally, we also use the (now standard) comparison principle for viscosity subsolutions and supersolutions of the eikonal equation, that can be found in [85].

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